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ON THE SHAPE OF THE TRAJECTORY OF THE MOTION OF A PARTICLE IN THE THREE DIMENSIONAL GENERAL INNER PRODUCT SPACE

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ABSTRACT

In this paper we are going to establish some necessary and sufficient conditions for the shapes of the smooth trajectories of the motion of the moving particles in the three dimensional general inner product space.

KEY WORDS
Inner product; Isometry; Frenet-Serret formulas

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INTRODUCTION

The Frenet–Serret formulas describe the kinematic properties of a particle moving along a smooth trajectory in the three dimensional Euclidean space R^3, with usual inner product. More specifically, the formulas describe the derivatives of the tangent, normal, and binormal unit vectors in terms of each other.
In [1], using linear algebra, the Frenet–Serret formulas would be generalized to three dimensional Euclidean spaces with general inner product σ. In this paper, using the results obtained in mentioned reference, we would like to obtain some necessary and sufficient conditions for some kinds of smooth trajectories in terms of generalized Frenet- Serret formulas [1].
Note that a plane in the inner product space (R^3,σ) can be described as the union of all the perpendiculars to a given line at a given point. In vector language, the plane through p orthogonal to $q \neq 0$ consists of all points r in R^3 such that $\sigma(r-p,q)=0$.

BASIC PROPERTIES

In the following theorem, a necessary and sufficient condition that a unit-speed smooth trajectory lies in a plane, would be determined.

Theorem 2.1. Let β be a unit-speed smooth trajectory in (R^3,σ) with $\kappa > 0$. Then β is a plane smooth trajectory if and only if $\tau = 0$.

Proof. Suppose β is a plane smooth trajectory. Then by the above considerations, there exist points p and q such that $\sigma(\beta(s)-p,q)=0$ for all s. Differentiation yields $\sigma(\beta'(s),q)=\sigma(\beta''(s),q)=0$. Thus q is always σ orthogonal to $T=\beta'$ and $N=\kappa^{-1}\beta''$. But B is also σ orthogonal to T and N, so, since B has unit length, $B=\pm||q||^{-1}q$. Thus $B'=0$, and by definition $\tau=0$. Conversely, suppose $\tau=0$. Thus $B'=0$; that is, B is parallel and may thus be identified with a point of R^3. We assert that β lies in the plane through $\beta(0)$ orthogonal to B. To prove this, consider the real valued function $f(s)=\sigma(\beta(s)-\beta(0),B)$ for all s. Then $f'(s)=0$, but obviously, $f(0)=0$, so f is identically zero. Thus β lies entirely in this plane orthogonal to the (parallel) binormal of β.

A circle of center $p \in R^3$ and radius $a \geq 0$ in (R^3,σ) consist of all is the set of all $w \in R^3$ such that $(w-p,w-p)=a^2$. Let $\bar{\imath},\bar{\jmath} \in R^3$ are two σ orthonormal vectors. The equation of a circle of radius a in (R^3,σ) is given by $\beta(s)= a\cos(a^{-1}s)\bar{\imath}+a\sin(a^{-1}s)\bar{\jmath}$. Then $\beta''(s)=-a^{-1}\cos(a^{-1}s)\bar{\imath}-a^{-1}\sin(a^{-1}s)\bar{\jmath}$, $\kappa(s)=\sqrt{\sigma(\beta''(s),\beta''(s))}=a^{-1}$. Therefore, β is a unit-speed smooth trajectory with $\kappa=a^{-1}$, $N(s)=\cos(a^{-1}s)\bar{\imath}-\sin(a^{-1}s)\bar{\jmath}$ and a circle of radius a has curvature a^{-1}.

Furthermore, the formula given there for the principal normal shows that for a circle, N always points toward its center. This suggests how to prove the following converse.

Theorem 2.2. If β is a unit-speed smooth trajectory with constant curvature $\kappa > 0$ and torsion zero, then β is part of a circle in (R^3,σ) of radius κ^{-1}.

Proof. Since $\tau=0$, β is a plane smooth trajectory. What we must now show is that every point of β is at distance κ^{-1} from some fixed point, which will be the center of the circle. Consider the smooth trajectory $=\beta+\kappa^{-1}N$. Using the hypothesis on β, and a Frenet-Serret formula in (R^3,σ), we find $\gamma'=\beta'+$

$\kappa^{-1}N' = T + \kappa^{-1}(-\kappa T + \tau B) = 0$. Hence the smooth trajectory γ is constant; that is, $\beta(s) + \kappa^{-1}N(s)$ has the same value, say c, for all s. But the distance from c to $\beta(s)$ is

$$d(c, \beta(s)) = ||c - \beta(s)|| = \sqrt{\sigma(c - \beta(s), c - \beta(s))} = \sqrt{\sigma(\kappa^{-1}N(s), \kappa^{-1}N(s))} = \kappa^{-1}.$$

In principle, every geometric problem about smooth trajectories in (R^3, σ) can be solved by means of the Frenet-Serret formulas corresponding to σ. In simple cases it may be just enough to record the data of the problem in convenient form, differentiate, and use the Frenet-Serret formulas in (R^3, σ). For example, suppose β is a unit-speed smooth trajectory that lies entirely in the sphere S of radius a centered at $c \in R^3$.

To stay in the sphere, β must smooth trajectory; in fact it is a reasonable guess that the minimum possible curvature occurs when β is on a great circle of S. Such a circle has radius a, so we conjecture that:

Theorem 2.3. A unit-speed spherical smooth trajectory β has curvature $\kappa \geq a^{-1}$, where a is the radius of its sphere in (R^3, σ).

Proof. Since every point of has distance a from c, the center of the sphere, we have $\sigma(\beta - c, \beta - c) = a^2$. Differentiation yields $\sigma(\beta - c, \beta') = 0$, that is, $\sigma(\beta - c, T) = 0$. Another differentiation gives $\sigma(\beta', T) + \sigma(\beta - c, T') = 0$, and by using a Frenet-Serret formula we get $\sigma(T, T) + \sigma(\beta - c, \kappa N) = 0$; hence $\kappa\sigma(\beta - c, N)\ ) = -1$. By the Schwarz inequality, $\kappa^{-1} = |\sigma(\beta - c, N)| \leq ||\beta - c|| \cdot ||N|| = a$ and we obtain the required result.

Continuation of this procedure leads to a necessary and sufficient condition expressed in terms of curvature and torsion for a smooth trajectory to be spherical, that is, lie on some sphere in (R^3, σ).

Theorem 2.4. Let β be a unit-speed smooth trajectory with $\kappa > 0, \tau \neq 0$ in (R^3, σ). (a) If β lies on a sphere of center c and radius a, then $\beta - c = -\kappa^{-1}N - (\kappa^{-1})'\tau^{-1}B$, and $a^2 = \kappa^{-2} + [(\kappa^{-1})'\tau^{-1}]^2$. Conversely, if $a^2 = \kappa^{-2} + [(\kappa^{-1})'\tau^{-1}]^2$ and $(\kappa^{-1})' \neq 0$, then β lies on a sphere of radius a.

Proof. As in the proof of Theorem 2.3, $\sigma(\beta - c, T) = 0, \sigma(\beta - c, N) = -\kappa^{-1}$. Again differentiation using Frenet-Serret formulas yields $\sigma(\beta', N) + \sigma(\beta - c, -\kappa T + \tau B) = -(\kappa^{-1})'$, or $\sigma(\beta - c, B) = -(\kappa^{-1})'\tau^{-1}$. Since $\beta - c = \sigma(\beta - c, T)T + \sigma(\beta - c, N)N + \sigma(\beta - c, B)B$, we have $\beta - c = -\kappa^{-1}N - (\kappa^{-1})'\tau^{-1}B$. But N and and B are σ orthonormal, so

$$a^2 = ||\beta - c||^2 = ||-\kappa^{-1}N - (\kappa^{-1})'\tau^{-1}B||^2 = \sigma(-\kappa^{-1}N - (\kappa^{-1})'\tau^{-1}B, -\kappa^{-1}N - (\kappa^{-1})'\tau^{-1}B) = \kappa^{-2} + [(\kappa^{-1})'\tau^{-1}]^2.$$

Conversely, let $a^2 = \kappa^{-2} + [(\kappa^{-1})'\tau^{-1}]^2$ and $(\kappa^{-1})' \neq 0$, differentiation yields $\kappa^{-1}\tau + \tau^{-1}(\kappa^{-1})'' + (\kappa^{-1})'(\tau^{-1})' = 0$. Define $\gamma = \beta + \kappa^{-1}N + (\kappa^{-1})'\tau^{-1}B$, a computation using preceding results yields $\gamma' = 0$, therefore $\beta + \kappa^{-1}N + (\kappa^{-1})'\tau^{-1}B = c$ for some constant $c \in R^3$.

Theorem 2.5. If δ be the spherical image of the unit-speed smooth trajectory β in (R^3, σ), then $\kappa_\delta \geq \kappa_\beta$.

Proof. The spherical image of a unit-speed smooth trajectory β in (R^3, σ) is the smooth trajectory $\delta = T_\beta = \beta'$ with the same Euclidean coordinates. Thus δ, lies on the unit sphere S, and the motion of δ represents the turning of β. The equation of δ implies that $T_\delta = \delta' = T_\beta{}' = \kappa_\beta N_\beta$, so the speed of δ is equal to the curvature of β, i.e., $v_\delta = \kappa_\beta$. Moreover,$T_\delta' = \delta'' = T_\beta{}'' = (\kappa_\beta N_\beta)' = \kappa_\beta' N_\beta + \kappa_\beta N_\beta' = \kappa_\beta' N_\beta + \kappa_\beta(-\kappa_\beta T_\beta + \tau_\beta B_\beta) = -\kappa_\beta^2 T_\beta + \kappa_\beta' N_\beta + \kappa_\beta \tau_\beta B_\beta$. So $T_\delta' = \kappa_\delta v_\delta N_\delta$ implies that $\kappa_\beta^4 + (\kappa_\beta')^2 + \kappa_\beta^2 \tau_\beta^2 = \kappa_\delta^2 \kappa_\beta^2$, and $\kappa_\delta \geq \kappa_\beta$.

Definition 2.6. A unit-speed smooth trajectory β in (R^3, σ) is a helix provided the unit tangent T of β has non zero constant value with some fixed σ unit vector.

Theorem 2.7. A unit-speed smooth trajectory β with $\kappa > 0$ in (R^3, σ) is a helix if and only if the ratio $\kappa^{-1}\tau$ is non zero constant.

Proof. It suffices to consider the case where α has unit speed. If α is a helix with $\sigma(T(s), U) = c \neq 0$, then $0 = \sigma(T'(s), U) = k\sigma(N, U)$. Since $\kappa > 0$, we conclude that $\sigma(N(s), U) = 0$. Thus for each s, U lies in the plane determined by T(s) and B(s). Orthonormal expansion yields $U = \sigma(U, T)T + \sigma(U, B)B$ and $\sigma(U, T)^2 + \sigma(U, B)^2 = 1$, so $\sigma(U, B)$ is also constant. By differentiating and applying Frenet-Serret formulas in (R^3, σ), we obtain $0 = \sigma(U, T)T' + \sigma(U, B)B' = (\kappa\sigma(U, T) - \tau\sigma(U, B))N$. Hence $\kappa\sigma(U, T) = \tau\sigma(U, B)$, so that $\kappa^{-1}\tau$ has non zero constant value $\sigma(U, B)^{-1}\sigma(U, T)$. Conversely, suppose that $\kappa^{-1}\tau$ is non zero constant. If $V = \kappa^{-1}\tau\, T + B$ we find $||V|| = \sqrt{1 + \kappa^{-2}\tau^2} > 1$ and $V' = \kappa^{-1}\tau\, T' + B' = 0$. This parallel vector field V determines a unit vector $U = ||V||^{-1}V$, such that $\sigma(U, T) = \sigma(||V||^{-1}V, T) = ||V||^{-1}\sigma(V, T)$=$(\kappa^{-1}\tau)\sqrt{(1 + \kappa^{-2}\tau^2)^{-1}}$, so β is a helix.

GEOMETRY OF THE GENERAL INNER PRODUCT SPACES

We recall some familiar features of plane geometry. First of all, two triangles are congruent if there is a rigid motion of the plane that carries one triangle exactly onto the other. Corresponding angles of congruent triangles are equal, corresponding sides have the same length; the areas enclosed are equal, and so on. Indeed, any geometric property of a given triangle is automatically shared by every congruent triangle.
Conversely, there are a number of simple ways in which one can decide whether two given triangles are congruent, for example, if for each the same three numbers occur as lengths of sides. In this section we shall investigate the isometries of the inner product space, and see how these remarks about triangles can be extended to other geometric objects in (R^3,σ).

The inner product of points $p=(p_1,p_2,p_3)$ and $q=(q_1,q_2,q_3)$ in R^3 is a number $\sigma(p,q)$ with the following three properties: (1) Bilinearity: $\sigma(ap+bq,r)=a\,\sigma(p,r)+b\sigma(q,r)$, $\sigma(r,ap+bq)=a\,\sigma(r,p)+b\sigma(r,q)$, for p,q and r are arbitrary points of R^3, and a and b are number (2) Symmetry: $\sigma(p,q)=(q,p)$, for arbitrary points p,q of R^3(3) Positive definiteness: $\sigma(p,p)\ \geq 0$, and $\sigma(p,p)\ =0$ if and only if $p=0$.

The norm of a point $p\in R^3$ is the number $||p||=\sqrt{\sigma(p,p)}$. The norm is thus a real-valued function on R^3, it has the fundamental properties $\left||p+q|\right|\leq\left||p|\right|+\left||q|\right|$ and $||ap||$= $|a|\ ||p||$ where $|a|$ is the absolute value of the number a. The distance between two points p and q in R^3 is defined by $d(p,q)=||p-q||$ [2].

An isometry of the inner product space is a mapping that preserves the distance d between points, i.e., a mapping $F: R^3\to R^3$ such that $d(p,q)=d(F(p),F(q))$ for all points p,q in R^3. An orthogonal transformation of R^3, is a linear transformation $C: R^3\to R^3$ that preserves inner products in the sense that $\sigma(C(p),C(q))=\sigma(p,q)$ for all p,q in R^3. It can be seen easily that an orthogonal transformation is an isometry of R^3. Moreover, If F is an isometry of R^3 such that $F(0)=0$, then F is an orthogonal transformation.

If F is an isometry of R^3, then there exists a unique translation T and a unique orthogonal transformation C such that $F=TC$ [3]. So, if T is translation by $a=(a_1,a_2,a_3)$, then $q=F(p)$ means $q=a+C(p)$, where by a standard result of linear algebra, a linear transformation of $C: R^3\to R^3$ is orthogonal if and only if its matrix, $[c_{ij}]$, is (σ) orthogonal with respect to a σ orthonormal basis [4].

Theorem 3.1. Let $F: R^3\to R^3$ be an isometry in (R^3,σ) with orthogonal part C. If $\beta=F\circ\alpha$, then $\beta'=C\alpha'$, $\beta''=C\alpha''$, $\beta'''=C\alpha'''$. Moreover, $\alpha',\alpha'',\alpha'''$ and β',β'',β''' are simultaneously linearly dependent.

Proof. If T is translation by a, then $\beta(t)=a+(C\circ\alpha)(t)$, now linearity of C and chain rule of differentiation [5] implies the theorem. The proof of the rest of the theorem is straightforward.

Theorem 3.2. Let $F: R^3\to R^3$ be an isometry and β be a unit-speed smooth trajectory in (R^3,σ), then $\bar{\beta}=F\beta$ is a unit-speed smooth trajectory in (R^3,σ).

Proof. Let C be be the σ orthogonal part of F. Then according to theorem 3.1, $\sigma(\bar{\beta}',\bar{\beta}')=\sigma(C\beta',C\beta')=\sigma(\beta',\beta')=1$.

Theorem 3.3. Let $F: R^3\to R^3$ be an isometry with orthogonal part C. Let β be a unit-speed smooth trajectory in (R^3,σ) with positive curvature, and let $\gamma=F\beta$. Then $\kappa_\gamma=\kappa_\beta,\tau_\gamma=\tau_\beta,T_\gamma=CT_\beta,N_\gamma=CN_\beta,B_\gamma=CB_\beta$.

Proof. Theorem 3.1 asserts that $T_\gamma=\gamma'=C\beta'=CT_\beta$. Therefore $\kappa_\gamma=\left||T_\gamma'|\right|=||\gamma''||=\left||\,C\beta''|\right|=\left||\,C\,T_\beta'|\right|=\left||C(\kappa_\beta N_\beta)|\right|=\kappa_\beta\left||CN_\beta|\right|=\kappa_\beta$. For finding B_γ, as in [1] let $\bar{B}_\gamma=\gamma'''-\sigma(\gamma',\gamma''')\gamma'-\kappa_\gamma{}^{-2}\sigma(\gamma'',\gamma''')\gamma''$, then a computation using Theorem 3.1 yields
$\overline{B_\gamma}=\ C\,\beta'''-\sigma(C\beta',C\beta''')C\beta'-\kappa_\gamma{}^{-2}\sigma(C\beta'',C\beta''')C\beta''=C(\beta'''-\sigma(\beta',\beta''')\beta'-\kappa_\beta{}^{-2}\sigma(\beta'',\beta''')\beta'')=C\overline{B_\beta}$.
Thus $B_\gamma=||\overline{B_\gamma}||^{-1}\overline{B_\gamma}=||C\overline{B_\beta}\,||^{-1}C\overline{B_\beta}\ =CB_\beta$. We can now find the torsion function τ_γ of the smooth trajectory γ as defined in [1] to be the real-valued map such that $B_\gamma{}'=-\tau_\gamma N_\gamma$. As a result $C(-\tau_\beta N_\beta)=CB_\beta{}'=(CB_\beta)'=B_\gamma{}'=-\tau_\gamma N_\gamma=-\tau_\gamma CN_\beta=C(-\tau_\gamma N_\beta)$ and so $\tau_\gamma=\tau_\beta$.

Smooth trajectories whose congruence is established by a translation are said to be parallel. Thus, smooth trajectories $\alpha,\beta: I\to R^3$ are parallel if and only if there is a point p in R^3 such that $\beta(s)=\alpha(s)+p$ for all s in I.

Remark 3.4. An argument, using elementary calculus shows that two smooth trajectories $\alpha,\beta: I\to R^3$ are parallel if their velocity vectors $\alpha'(s)$ and $\beta'(s)$ are parallel for each s in I. In this case, if $\alpha'(s_0)=\beta'(s_0)$ for some s_0 in I, then $\alpha=\beta$.

Definition 3.5. Two smooth trajectories $\alpha, \beta: I \to R^3$are congruent provided there exists an isometry $F: R^3 \to R^3$ such that $\beta = F\alpha$.

The following theorem is an important converse of Theorem 3.3. The proof is parallel to a similar theorem in $(R^3, <,>)$ [6].

Theorem 3.6. If $\alpha, \beta: I \to R^3$are unit-speed smooth trajectories with same torsions and same positive curvatures in (R^3, σ), then α and β are congruent.

Proof. First of all, note that if two frames on R^3, say u_1, u_2, u_3 at the point p and v_1, v_2, v_3 at the point q, there exists a unique isometry $F: R^3 \to R^3$with orthogonal part C such that $C(u_i) = v_i$ for $i = 1,2,3$ [4]. Then consider a number, say t_0, in the interval I. Let F be the isometry that carries the Frenet-Serret frame $T_\alpha(t_0), N_\alpha(t_0), B_\alpha(t_0)$ of α at $\alpha(t_0)$ to the Frenet-Serret frame $T_\beta(t_0), N_\beta(t_0), B_\beta(t_0)$ of β at $\beta(t_0)$. Denote the Frenet-Serret apparatus of $\overline{\alpha} = F\alpha$ by $\overline{\kappa}, \overline{\tau}, \overline{T}, \overline{N}, \overline{B}$, then it follows immediately from Theorem 3.3 and the above considerations that $\overline{\alpha}(t_0) = \beta(t_0)$, $\overline{T}(t_0) = T_\beta(t_0)$, $\overline{N}(t_0) = N_\beta(t_0)$, $\overline{B}(t_0) = T_\beta(t_0)$, $\overline{\kappa} = \kappa_\beta, \overline{\tau} = \tau_\beta$. Now since $\overline{\alpha}(t_0) = \beta(t_0)$, it will follow that $F\circ\alpha = \beta$. On the interval I, consider the real-valued function $f = \sigma(\overline{T}, T_\beta) + \sigma(\overline{N}, N_\beta) + \sigma(\overline{B}, B_\beta)$. Since these are unit vector fields, the Schwarz inequality shows that $f \le 3$. Above considerations also imply that $f(0) = 3$. Now consider $f' = \sigma(\overline{T}', T_\beta) + \sigma(\overline{T}, T_\beta') + \sigma(\overline{N}', N_\beta) + \sigma(\overline{N}, N_\beta') + \sigma(\overline{B}', B_\beta) + \sigma(\overline{B}, B_\beta')$, Substitute the Frenet-Serret formulas in (R^3, σ), in this expression and use the equations $\overline{\kappa} = \kappa_\beta, \overline{\tau} = \tau_\beta$ implies that

$$f' = \overline{\kappa}\sigma(\overline{N}, T_\beta) + \kappa_\beta\sigma(\overline{T}, N_\beta) - \overline{\kappa}\sigma(\overline{T}, N_\beta) + \overline{\tau}\sigma(\overline{B}, N_\beta) - \kappa_\beta\sigma(\overline{N}, T_\beta) + \tau_\beta\sigma(\overline{N}, B_\beta) - \overline{\tau}\sigma(\overline{N}, B_\beta) - \tau_\beta\sigma(\overline{B}, N_\beta) = 0.$$

Thus $f = 3$ and therefore $\sigma(\overline{T}, T_\beta) = 1$. This completes the proof.

CONFLICT OF INTEREST

There is no conflict of interest.

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