IIOAB JOURNAL
ISSN: 0976-3104

ARTICLE

MOVING FRAME EQUATIONS IN THE THREE DIMENSIONAL GENERAL INNER PRODUCT SPACE

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ABSTRACT

In this paper, we are going to generalize the Frenet-Serret formulas for the moving frames in the three dimensional space R^3, in the case that the whole space admits the general form of inner product.

KEY WORDS

Covariant derivative; Moving frame; Vector field

Published: 14 October 2016

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INTRODUCTION

Vectors are used widely in physics and engineering to describe forces, velocities, angular momentum, and many other concepts. To obtain a definition that is both practical and precise, we shall describe an "arrow" in R^3 by giving its starting point p and the change, or vector v, necessary to reach its end point $p+v$. Strictly speaking, v is just a point of R^3. A tangent vector v_p to R^3 consists of two points of R^3, its vector part v and its point of application p.

A vector field V on R^3 is a map that assigns to each point p of R^3 a tangent vector $V(p)$ to R^3 at p. There is a natural algebra of vector fields. At each point p, the values $V(p)$ and $W(p)$ are in the same vector space, the tangent space T_pR^3, consequently, the formula for the addition is the same as for addition of maps, $(V+W)(p) = V(p) + W(p)$ or all $p \in R^3$.

If f is a real-valued map on R^3 and V is a vector field on R^3, then fV is defined to be the vector field on R^3 such that $(fV)(p) = f(p)V(p)$ for all $p \in R^3$. Let f be a differentiable real-valued map on R^3, and let $v_p$be a tangent vector to R^3, then the number $v_p[f] = \frac{d}{dt}f(p+tv)_{|t=0}$ is called the derivative of f with respect to v_p. It can be seen that, if $v_p = (v_1, v_2, v_3)_p$ is a tangent vector to R^3, then $v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)$ [1].

If f and g be maps on R^3, v_p and w_p tangent vectors, a and b numbers, then

$$(av_p + bw_p)[f] = av_p[f] + bw_p[f],\ v_p[af+bg] = av_p[f] + bv_p[g],\ v_p[fg] = v_p[f]g(p) + f(p)v_p[g].$$

For a vector field V, $V[f]$ is the real-valued map whose value at each point p is the number $V(p)[f]$. Similarly, if f, g, h are real-valued maps $V, W,$ are vector fields on R^3, and $a, b \in R$, then

$$(fV+gW)[h] = fV[h] + gW[h],\ V[af+bg] = aV[f] + bV[g], V[fg] = V[f]g + fV[g].$$

Replacing f by a vector field W on R^3 gives a vector field $t \to W(p+tv)$ on the smooth trajectory $t \to p + tv$. Then the derivative of W with respect to v will be the derivative of $t \to W(p+tv)$ at $t=0$. In fact, if W be a vector field on R^3, and v be a tangent vector field to R^3 at the point p, then the covariant derivative of W with respect to v is the tangent vector $\nabla_v W(p) = \frac{d}{dt}W(p+tv)_{|t=0}$. Evidently $\nabla_v W(p)$ measures the initial rate of change of $W(p)$ as p moves in the v direction. If U_1, U_2, and U_3 be the vector fields on R^3 and $W = \sum w_i U_i$ is a vector field on R^3, and v is a tangent vector at p, then $\nabla_v W(p) = \sum v_p[w_i]U_i\,(p)$.

Moreover, if v and w be tangent vectors to R^3 at p, and let Y and Z be vector fields on the general inner product space (R^3, σ), then for numbers a, b and map f, we have

$\nabla_{(av+bw)}Y(p) = a\nabla_v Y(p) + b\nabla_w Y(p),\ \ \nabla_v(aY + bZ)(p) = a\nabla_v Y(p) + b\nabla_v Z(p), \nabla_v(fY)(p) = v_p[f]Y(p) + f(p)\nabla_v Y(p), v_p[\sigma(Y,Z)] = \sigma\big(\nabla_v Y(p), Z(p)\big) + \sigma(Y(p), \nabla_v Z(p))$.

Using the point wise principle, we can take the covariant derivative of a vector field W with respect to a vector field V, rather than a single tangent vector v. The result is the vector field $\nabla_V W$ whose value at each point p is $\nabla_{V(p)} W$.

It follows immediately from above considerations that if $W = \sum w_i U_i$, then $\nabla_V W = \sum V[w_i]U_i$. If f, g be differentiable maps, $a, b \in R$ and V, W, Y, and Z be vector fields on R^3, then as a result of the preceding identities we have, $\nabla_{(fV+gW)}Y = f\nabla_V Y + g\nabla_W Y,\ \nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z,\ \ \nabla_V(fY) = V[f]Y + f\nabla_V Y, V(\sigma(Y,Z)) = \sigma(\nabla_V Y, Z) + \sigma(Y, \nabla_V Z)$ [1].

Frenet- Serret's essential idea was very simple: To each point of a smooth trajectory, a frame is assigned, then using orthonormal expansion expresses the rate of change of the frame in terms of the frame itself. This, of course, is just what the Frenet- Serret formulas do in the case of a smooth trajectory [1, 2].

In the next, we shall carry out this scheme for the inner product space (R^3,σ). We shall see that geometry of smooth trajectories and surfaces in (R^3,σ) is not merely an analogue, but actually a corollary, of these basic results [2].

THE METHOD OF MOVING FRAMES

If $A(p)=(p,a(p)$ and $B(p)=(p,b(p))$ are vector fields on R^3, then the inner product $\sigma(A,B)$ of A and B is the differentiable real-valued map on R whose value at each point p is $\sigma(a(p),b(p))$. The norm $||A||$ of A is the real-valued map on R^3 whose value at p is $||a(p)||$.
Vector fields T_1,T_2,T_3 on (R^3,σ) constitute a frame field on (R^3,σ) provided where $\sigma(T_i,T_j)=\delta_{ij}(1\leq i,j\leq 3)$ where δ_{ij} is the Kronecker delta [3].

The following useful result is an immediate consequence of orthonormal expansion [3].

Lemma 3.1. Let T_1,T_2,T_3 be a frame field on (R^3,σ). If V is a vector field on R^3, then $V=\sum f_iT_i$, where the maps $f_i=\sigma(V,T_i)$. If $V=\sum f_iT_i$ and $W=\sum g_iT_i$, then $\sigma(V,W)=\sum f_ig_i$. In particular, $\left||V|\right|=\sqrt{\sum f_i^2}$.

THE COVARIANT DERIVATIVE OF A FREME FIELD

The Frenet- Serret formulas express the derivatives of *T*, *N*, *B* in terms of *T*, *N*, *B*, and thereby define curvature and torsion. We shall now do the same thing with an arbitrary frame field T_1,T_2,T_3 on (R^3,σ), namely, express the covariant derivatives of these vector fields in terms of the vector fields themselves. We begin with the covariant derivative with respect to an arbitrary tangent vector v at a point p. Then $\nabla_vT_i=\sum c_{ij}(v)T_j(p)$ for $i,j=1,2,3$. By orthonormal expansion the coefficients of these equations are $c_{ij}(v)=\sigma(\nabla_vT_i,T_j(p))$. A 1-form f on R^3 is a real-valued map on the set of all tangent vectors to R^3such that f is linear at each point [4].

Lemma 4.1. Let T_1,T_2,T_3 be a frame field on (R^3,σ). For each tangent vector v to R^3 at the point p, let $c_{ij}(v)$ be defined as above. Then each c_{ij} is a 1-form, and $c_{ij}+c_{ji}=0$.

Proof. By definition, c_{ij} is a real-valued map on tangent vectors, so to verify that c_{ij} is a 1-form, it suffices to check the linearity condition. Using above considerations, we get,

$$\begin{aligned}c_{ij}(av+bw)&=\sigma\left(\nabla_{av+bw}T_i,T_j(p)\right)=\sigma(a\nabla_vT_i+b\nabla_wT_i,T_j(p))\\&=a\sigma(\nabla_vT_i,T_j(p))+b\sigma(\nabla_wT_i,T_j(p))=\ ac_{ij}(v)+b\,c_{ij}(w).\end{aligned}$$

To prove that $c_{ij}+c_{ji}=0$ we show that $c_{ij}(v)+c_{ji}(v)=0$ for every tangent vector v. By definition of frame field, $\sigma(T_i,T_j)=\delta_{ij}$, and since each Kronecker delta has constant value 0 or 1, then $v[\sigma(T_i,T_j)]=0$, so the above considerations yields $\sigma\left(\nabla_vT_i,T_j(p)\right)+\sigma(T_i(p),\nabla_vT_j)=0$ and the proof is complete [1, 5].

The definition $c_{ij}(v)=\sigma(\nabla_vT_i,T_j(p))$ shows that $c_{ij}(v)$ is the initial rate at which T_i rotates toward T_j as p moves in the v direction. Thus the 1-forms c_{ij} contain this information for all tangent vectors to R^3.

Theorem 4.2. For any vector field V on (R^3,σ), $\nabla_VT_i=\sum c_{ij}(V)\,T_j(1\leq i,j\leq 3)$. In expanded form, $\nabla_VT_1=c_{12}(V)T_2+c_{13}(V)T_3, \nabla_VT_2=-c_{12}(V)T_1+c_{23}(V)T_3,\ \nabla_VT_3=-c_{13}(V)T_1-c_{23}(V)T_2.$

Proof. Let i be fixed and $p\in R^3$, then according to the previous considerations $\nabla_{V(p)}T_i=\sum c_{ij}(V(p))T_j(p)$, so $\nabla_VT_i=\sum c_{ij}(V)\,T_j$. When $i=j$, the skew-symmetry condition $c_{ij}+c_{ji}=0$ becomes $c_{ii}=0$ for i =1,2,3. Hence this condition has the effect of reducing the nine 1-forms c_{ij} for $1\leq i,j\leq 3$ to essentially only three, say c_{12},c_{13},c_{23}.
Thus in expanded form, the equations in Theorem 3.2, called moving frame equations, become $\nabla_VT_1=c_{12}(V)T_2+c_{13}(V)T_3, \nabla_VT_2=-c_{12}(V)T_1+c_{23}(V)T_3,\ \nabla_VT_3=-c_{13}(V)T_1-c_{23}(V)T_2.$

These equations play a fundamental role in all the differential geometry of (R^3,σ). The following theorem explains that they are generalized form of Frenet-Serret formulas,

Theorem 4.3. Let β be a unit-speed smooth trajectory in (R^3,σ) with $\kappa>0$, and suppose that T_1,T_2,T_3 is a frame field on (R^3,σ) such that the restriction of these vector fields to β gives the generalized Frenet-Serret frame field T,N,B of β. Then $c_{12}(T)=\kappa,\ c_{13}(T)=0,\ c_{23}(T)=\tau.$

Proof. First of all not that, if W be a vector field defined on a region containing a regular smooth trajectory α, then $W_\alpha:t\to W(\alpha(t))$, the vector field on α, satisfies $\nabla_{\alpha'(t)}W=(W_\alpha)'(t)$. Thus using the generalized Frenet-Serret formulas in [2] implies that

$$c_{12}(T)=\sigma(\nabla_T T,N(p))=\sigma\left(\nabla_{\beta'(t)}T,N(p)\right)=\sigma((T_\beta)'(t),N(p))$$
$$=\sigma(\kappa N(p),N(p))=\kappa,$$
$$c_{13}(T)=\sigma(\nabla_T B,B(p))=\sigma(\nabla_{\beta'(t)}B,B(p))=\sigma((B_\beta)'(t),B(p))$$
$$=\sigma(-\tau N(p),B(p))=0,$$
$$c_{23}(T)=\sigma(\nabla_T N,B(p))=\sigma(\nabla_{\beta'(t)}N,B(p))=\sigma((N_\beta)'(t),B(p))$$
$$=\sigma(-\kappa T(p)+\tau B(p),B(p))=\tau.$$

Corollary 4.4. Let β be a unit-speed smooth trajectory in (R^3,σ) with $\kappa>0$, and suppose that T_1,T_2,T_3 is a frame field on (R^3,σ) such that the restriction of these vector fields to b gives the General Frenet- Serret frame field T,N,B of β corresponding to σ. Then $\nabla_T T=\kappa N, \nabla_T N=-\kappa T+\tau B, \nabla_T B=-\tau B$.

Since each regular smooth trajectory in (R^3,σ) has a unit speed reparametrization [2], we have,

Corollary 4.5. Let β be a smooth trajectory in (R^3,σ) with $\kappa>0$, and suppose that T_1,T_2,T_3 is a frame field on (R^3,σ) such that the restriction of these vector fields to b gives the general Frenet- Serret frame field T,N,B of β corresponding to σ. Then $\nabla_T T=\kappa v N, \nabla_T N=-\kappa v T+\tau v B, \nabla_T B=-\tau v B$.

APPLICATIONS TO GRAVITATIONAL FIELD

Let r,θ,z be the usual cylindrical coordinate maps on R^3. We shall pick a unit vector field in the direction in which each coordinate increases when the other two are held constant. For r this is evidently $T_1=(x\sqrt{(x^2+y^2)^{-1}},y\sqrt{(x^2+y^2)^{-1}},0)$ pointing straight out from the z axis. Then $T_2=(-y\sqrt{(x^2+y^2)^{-1}},x\sqrt{(x^2+y^2)^{-1}},0)$ points in the direction of increasing θ. Finally, the direction of z is, of course, straight up, so $T_3=(0,0,\frac{1}{\sqrt{2}})$. It is easy to check that $\sigma(T_i,T_j)=\delta_{ij}$ for σ defined as in [2], so this is a frame field defined on all of R^3 except the z axis. We call it the cylindrical frame field on R^3 [6].

For an arbitrary differentiable vector field V, a computation yields

$$\nabla_V T_1=\left(-y\sqrt{(x^2+y^2)^{-1}}v_1+x\sqrt{(x^2+y^2)^{-1}}v_2\right)T_2,\ \nabla_V T_2=\left(y\sqrt{(x^2+y^2)^{-1}}v_1-x\sqrt{(x^2+y^2)^{-1}}v_2\right)T_1,$$
$$\nabla_V T_3=0.$$

In a similar way, a frame field J_1, J_2, J_3 can be derived from the spherical coordinate functions ρ, θ, φ on R^3 [6]. The unit vector field J_1, in the direction of increasing ρ, points straight out from the origin; hence it can be expressed as

$$J_1=(x\sqrt{(x^2+y^2+z^2)^{-1}},y\sqrt{(x^2+y^2+z^2)^{-1}},z\sqrt{(x^2+y^2+z^2)^{-1}}).$$

Similarly, the vector field for increasing θ and φ are

$$J_2=(-y\sqrt{(x^2+y^2)^{-1}},x\sqrt{(x^2+y^2)^{-1}},0)\ ,$$
$$J_3=$$
$$(-xz\sqrt{(x^2+y^2)^{-1}(x^2+y^2+z^2)^{-1}},-yz\sqrt{(x^2+y^2)^{-1}(x^2+y^2+z^2)^{-1}},\sqrt{(x^2+y^2)[2(x^2+y^2+z^2)]^{-1}}),$$

respectively.

By repeated use of the fundamental identity in trigonometry, we check that J_1, J_2, J_3 is a frame field in (R^3,σ), called the spherical frame field on R^3, in which σ is defined as in [2]. Its actual domain of definition is R^3 minus the z axis, as in the cylindrical case.

Newton's law of gravitation states that a body of mass m_1 exerts a force on a body of mass m_2. The magnitude of the force is $Gm_1m_2r^{-2}$, where r is the distance between their centers of gravity and G is a constant. The direction of the force on m_2 is from m_2 to m_1. Thus if m_1 lies at the origin of S, and m_2 lies at $x\in S$, the force on m_2 is $-Gm_1m_2\sqrt{(x^2+y^2+z^2)^{-3}}(x,y,z)$. We must now face the fact that both bodies will move. However, if m_1 is much greater than m_2, its motion will be much less since acceleration is inversely proportional to mass.

We therefore make the simplifying assumption that one of the bodies does not move, in the case of planetary motion, of course it is the sun that is assumed at rest. One might also proceed by taking the center of mass at the origin, without making this simplifying assumption. Let now the sun is at the origin of S and consider the vector field corresponding to a planet of given mass m.

This field is then $V(x,y,z)=-c\sqrt{(x^2+y^2+z^2)^{-3}}(x,y,z)$ where c is a non-zero constant. Therefore, $\nabla_V T_1=\nabla_V T_2=\nabla_V T_3=0$. i.e., the initial rate at which each unit vector field in cylindrical coordinate rotates toward another one, as point moves in the direction of gravitational field in (R^3,σ), is zero. For an arbitrary differentiable vector field V, a similar computation yields, $c_{23}=-yz\sqrt{(x^2+y^2+z^2)^{-1}}v_1+xz\sqrt{(x^2+y^2+z^2)^{-1}}v_2$. Therefore, for the gravitational field $c_{23}=0$, i.e., the initial rate at which the unit vector field J_2 in spherical coordinate rotates toward J_3, as point moves in the direction of gravitational field in (R^3,σ), is zero.

IIOAB JOURNAL
ISSN: 0976-3104

CONFLICT OF INTEREST

There is no conflict of interest.

ACKNOWLEDGEMENTS

None

FINANCIAL DISCLOSURE

None

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