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ARTICLE

ON THE LIMIT PROPERTIES OF THE VERHULST POPULATIONS

Ali Parsian

Department of Mathematics, Tafresh University, Tafresh, IRAN

ABSTRACT

Those who have studied the population size, i.e., the demographers, are eager to build a real model which not only describe the demographic changes in the past periods, but can predict these changes in coming periods. They are very eager to find what factors are effective in the population size, and which of these factors can be controlled by social policies? These are difficult questions. The prediction of Verhulst law about great bunch of populations has been correct. In this study, we are going to show that the populations modeled by Verhulst equation are not a limit of a sequence of cyclic generations.

KEY WORDS
2-cycle; Vector space; Verhulst equation

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***Corresponding Author**
Email:
:Parsian@Tafreshu.ac.ir

INTRODUCTION

The logistic (Verhulst) growth model and its current applications in the problem of growth in biology are all based on the notion that the isothermal momentary growth rate is proportional to the momentary population's size and the fraction of resources that are still available in the habitat [1, 2].

A mathematical model, as a combination of probability and Verhulst population model indicating the effort of some believes in decreasing, constancy or increasing the population of an ethnic minority, or whole peoples with common believes of a country, in the crossing stage from one generation to another had been provided in [3].
In this study, assuming the constancy of two coefficients of Verhulst equation, we are going to find some properties of the acyclic populations which are a limit of a sequence of cyclic generations.

PRELIMINARIES

Let J be a compact interval and $k \geq 1$. The set of all maps $J \to J$ with k continuous derivatives will be denoted by $D^k(J)$. We define

$$\forall f, g \in D^k(J, R), \quad \forall \lambda \in R, \qquad \forall x \in J; \quad (f+g)(x) = f(x) + g(x), (\lambda f)(x) = \lambda f(x).$$

For $f \in D^k(J)$ let $\|f\| = \sup_{x \in J}(|f(x)| + |f'(x)| + \cdots + |f^{(k)}(x)|)$, then

The proofs of the following theorems are straightforward [4].

Theorem 2.1. $(D^k(J), +, .)$ is a vector space.

Theorem 2.2. $(D^k(J), \| \ \|)$ is a normed vector space.

MATERIALS AND METHODS

In the following, using the method similar to [5], we are going to investigate the properties of the populations which are a limit of a sequence of cyclic ones.

Theorem 3.1. Let $f, f_n \in D^k(J)$, $f_n \to f$ in $(D^k(J), \| \ \|)$, and $a_n, p \in J$ are such that $a_n \to p$. Then $f_n(a_n) \to f(p)$.

Proof. For $\varepsilon > 0$ there exists $n_0 \in N$ such that if $n > n_0$ then $\|f_n - f\| < \varepsilon/2$. Therefore $|f_n(a_n) - f(a_n)| \leq \|f_n - f\| < \varepsilon/2$.
Since f is continuous and $a_n \to p$, then there exists $n_1 \in N$ such that if $n > n_1$ then $|f(a_n) - f(p)| < \varepsilon/2$. Now if $n_2 = \max\{n_0, n_1\}$ and $n > n_2$ then $|f_n(a_n) - f(p)| \leq |f_n(a_n) - f(a_n)| + |f(a_n) - f(p)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Theorem 3.2. Let $f, f_n \in D^k(J)$, $f_n \to f$ in $(D^k(J), \| \ \|)$ and f is an acyclic map. For $n \in N$ let $\{a_n, b_n\}$ be a 2-cycle for f_n such that $a_n \to p \in J$, $b_n \to q \in J$. Then $p = q$ and p is a fixed point for f, and $f'(p) = -1$.

Proof. Since $f_n(a_n) = b_n$, $f_n(b_n) = a_n$, Theorem 3.2 asserts that $f(p) = q$, $f(q) = p$. Now acyclicity of f implies that $p = q$. Moreover, $f_n \to f$ in $(D^k(J), \| \ \|)$ asserts that $f_n' \to f'$, and so $-1 = (a_n - b_n)(b_n - a_n)^{-1} = (f_n(b_n) - f_n(a_n))(b_n - a_n)^{-1}$. Thus $f_n'(a_n) = -1$. On the other hand, $f_n \to f$ and an argument similar to the proof of Theorem 3.1 implies that $f'(p) = \lim_{n \to \infty} f_n'(a_n) = -1$.

The following theorem is a consequence of the Mean Value Theorem [6].

Theorem 3.3. Let $a < p < b$ are the fixed points of $f \in D^1(J)$. Then there are two different points $s, t \in J$ such that

$a < s < p < t < b$ and $f'(s) = f'(t) = 1$. Moreover, if $f \in D^2(J)$, then there exists some point $s < z < t$ such that $f'''(z) = 0$.

Theorem 3.4. Let $f, f_n \in D^k(J), k \geq 2, f_n \to f$ in $(D^k(J), \| \ \|)$ and f is an acyclic map. For $n \in N$, let $\{a_n, b_n\}$ be a 2-cycle for f_n such that $a_n \to p \in J$, $b_n \to q \in J$. Then $(f \circ f)''(p) = 1, (f \circ f)'''(p) = 0, f''(p) = 0$.

Proof. Without loss of generality, we can assume $\forall n \in N;\ a_n < b_n$. Since $f_n(a_n) = b_n,\ f_n(b_n) = a_n,$ so the fixed point theorem [6] implies that there exists $a_n < p_n < b_n$ such that $f_n(p_n) = p_n$. According to assumptions $a_n \to p$ and $b_n \to q$ and Theorem 3.2, we have $p = q$ and so $p_n \to p$. Therefore Theorem 3.1 asserts that $f_n(p_n) \to f(p)$ or $p_n \to f(p)$, and so $f(p) = p$. Moreover, a_n, p_n and b_n are the fixed points for $f_n \circ f_n$, because $f_n \circ f_n(a_n) = f_n(f_n(a_n)) = f_n(b_n) = a_n$, $f_n \circ f_n(b_n) = f_n(f_n(b_n)) = f_n(a_n) = b_n$ and $f_n \circ f_n(p_n) = f_n(f_n(p_n)) = f_n(p_n) = p_n$. Now Theorem 3.3 implies that there exist some points $a_n < s_n < p_n$, $p_n < t_n < b_n$ and $s_n < z_n < t_n$ such that $(f_n \circ f_n)'(s_n) = (f_n \circ f_n)'(t_n) = 1, (f_n \circ f_n)''(z_n) = 0$. But convergence in $(D^k(J), \| \ \|)$ implies that $0 = (f_n \circ f_n)'''(z_n) \to (f_n \circ f_n)'''(p)$, and so $(f \circ f)'''(p) = 0$. On the other hand, the chain rule of differentiation [6] and Theorem 3.2 implies $(f \circ f)''(p) = f'(f(p))f'(p) = f'(p)^2$=1, $0 = (f \circ f)'''(p) = 2f'(p)f''(p) = -2f''(p)$ and so $f''(p) = 0$.

DISCUSSION AND CONCLUSION

According to Verhulst theory, the law of the growth of population is of the form $f(x) = qx - rx^2$, in which $x, f(x), q$ and r denotes the population of the present generation, the population of the next generation, the rate of collaboration and the rate of competition respectively and vary from one society to another [7]. In the most today's societies, which the internal reduction factors of r and increasing factor of q still exist, the subject of population growth is very serious. A provided mathematical model shows that in addition to medical advances, health and nutrition, some ancient beliefs of peoples are effective on the increasing of the rate of collaboration and so in the growth of population. If the rate of collaboration is $1 \cdot 5$ and the number of initial population is bounded above by $(2r)^{-1}$, then the next generations of population increases and remains bounded by the same upper bound. In this case, if the number of initial population is bounded below by $(2r)^{-1}$, then the next generations of population decreases and remains bounded by the same lower bound. If the rate of collaboration is 2, the population increases but does not exceed from r^{-1}. If the rate of collaboration is 2.25 (res. 2.75), and if the number of population of a generation is less than $5(4r)^{-1}$ (res. $7(4r)^{-1}$), then it will be increasing in the next one, and if the number of population of a generation is more than $5(4r)^{-1}$ (res. $7(4r)^{-1}$), it will be decreasing in the next stage. Therefore the variation of the number of population will be alternative. In all of these cases the number of population is bounded above by $2r^{-1}$ [3].
Some of the most important properties for the populations which are modeled by Verhulst equation can be deduce from the preceding considerations. In fact, the non zero fixed point of Verhulst equation is $p = qr^{-1}$. A simple computation yields that $f'(p) = 2^{-1}q, f''(p) = -2r$, $(f \circ f)''(p) = 4^{-1}q^2$ and $(f \circ f)'''(p) = -2qr$. Therefore the quadruple system of equations obtained in Theorems 3.2 and 3.4 does not have an acceptable solution, i.e., the populations modeled by Verhulst equation are not a limit of a sequence of cyclic generations.

CONFLICT OF INTEREST

There is no conflict of interest.

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FINANCIAL DISCLOSURE

None

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