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ARTICLE

GEOMETRIZATION OF THE MOTION OF A PARTICLE IN THE THREE DIMENSIONAL GENERAL INNER PRODUCT SPACE

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ABSTRACT

The Frenet-Serret formulas describe the kinematic properties of a moving particle along a smooth trajectory in the three dimensional Euclidean space with standard inner product. More specially, the formulas describe the derivatives of tangent, normal and binormal unit vectors in terms of each other. The formulas are named after the two French mathematicians who independently discovered them, Jean Frédéric Frenet in his thesis in 1847 and Joseph Alfred Serret in 1851. Here, using linear algebra, the Frenet-Serret formulas would be generalized in the three dimensional general inner product space.

KEY WORDS
Inner product space; Frenet-Serret formulas; Gram-Schmitt process

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INTRODUCTION

The Frenet-Serret formulas describe the properties of the smooth trajectory of a moving particle in the three dimensional space R^3, in the case that the whole space admits the standard dot product [1].
In this study, using an orthonormal basis constructed by applying the Gram-Schmidt process in linear algebra, we are going to generalize the same formulas in the general case, i.e., the case that R^3 has the general inner product form σ given below [2].

We begin with some ingredients which are necessary for this purpose. The standard inner product of R^3 would be denoted by $<,>$, i.e., $<p,q>=p_1q_1+p_2q_2+p_3q_3$ for $p=(p_1,p_2,p_3)$ and $q=(q_1,q_2,q_3)$ in R^3. More generally, the inner product of points $p=(p_1,p_2,p_3)$ and $q=(q_1,q_2,q_3)$ in R^3 is a number $\sigma(p,q)$ with the following properties: (1) Bilinearity: $\sigma(ap+bq,r)=a\sigma(p,r)+b\sigma(q,r), \sigma(r,ap+bq)=a\sigma(r,p)+b\sigma(r,q)$ for arbitrary points p,q and $r$of R^3, and a,b are real numbers. (2) Symmetry: $\sigma(p,q)=\sigma(q,p)$. (3) Positive definiteness: $\sigma(p,p)\geq 0$ and $\sigma(p,p)=0$ if and only if $p=0$.

The norm of a point $p\in R^3$ is the number $||p||=\sqrt{\sigma(p,p)}$. The norm is thus a real-valued map on R^3, it has the fundamental properties $||p+q||\leq||p||+||q||$ and $||ap||=|a|\,||p||$ where $|a|$ is the absolute value of the number a [3]. If p and q are points of R^3, the distance from p to q is the number $||p-q||$.

Definition 1.1. Let $\alpha: I\to R^3$ be a smooth trajectory. If $h: J\to I$ is a differentiable map on an open interval J, then the composite map $\beta=\alpha(h): J\to R^3$ is a smooth trajectory called a reparametrization of α by h [4]. Using the chain rule for a composition of real-valued maps implies that $\acute{\beta}(s)=\acute{\alpha}(h(s))h'(s)$ [5].

Definition 1.2. Let $\alpha: I\to R^3$ be a smooth trajectory in R^3 with $\alpha=(\alpha_1,\alpha_2,\alpha_3)$. For each number t in *I*, the velocity vector of α at t is the tangent vector at the point $\alpha(t)$ in R^3. In terms of Euclidean coordinates $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, the velocity vector of α at t is $\acute{\alpha}(t)=(\alpha_1'(t),\alpha_2'(t),\alpha_3'(t))$. The speed map v of α is given by $v(t)=\sqrt{\sigma(\acute{\alpha}(t),\acute{\alpha}(t))}$.

From the viewpoint of calculus, the most important condition on a smooth trajectory α is that it be regular, that is, have all velocity vectors different from zero. The distance traveled by a moving point is determined by integrating its speed with respect to time. Thus, the arc length of α from $t=a$ to $t=b$ would be the number $l=\int_a^b||\acute{\alpha}(t)||dt$.

CHANGING THE PARAMETER

Substituting the formula for $||\acute{\alpha}||$ given above, we get the formula for arc length. This length involves only the restriction of α (defined on some open interval) to the closed interval $[a,b]$. Sometimes one is interested only in the route followed by a smooth trajectory and not in the particular speed at which it traverses its route. One way to ignore the speed of a smooth trajectory α is to reparametrize α to smooth trajectory β that has unit speed $||\beta'||=1$. Then β represents α "standard trip" along the route of α.

Theorem 2.1. If α is a regular smooth trajectory in (R^3,σ), then there exists a reparametrization β of α such that β has unit speed.

Proof. Let t_0 be an arbitrary number in the domain I of $\alpha: I\to R^3$, and consider the arc length map $s(t)=\int_{t_0}^t||\acute{\alpha}(u)||du$. Thus the derivative of the map $s=s(t)$ is the speed map $v=||\acute{\alpha}(t)||$ of α. Since α is regular, by definition $\acute{\alpha}$ is never zero; hence $s'(t)>0$. By the inverse mapping theorem [6], the map s has an inverse map $t=t(s)$, whose derivative at $s=s(t)$ is the reciprocal of at $t=t(s)$. In particular, $t'(s)>0$. Let β be the reparametrization $\beta(s)=\alpha(t(s))$ of α. We assert that β has unit speed. In fact, as asserted above $\acute{\beta}(s) = \acute{\alpha}(t(s))t'(s)$. Hence, by the preceding notes, the speed of β is $||\acute{\beta}(s)|| = ||\acute{\alpha}(t(s))||\, t'(s) = s'(t(s))t'(s) = 1$.

THE GENERAL FRENET-SERRET FORMULAS

We now derive mathematical measurements of the turning and twisting of a smooth trajectory in (R^3, σ). Throughout this section we deal only with unit-speed trajectories, in the next we extend the results to arbitrary regular trajectories.

Let $\beta: I \to R^3$ be a unit-speed smooth trajectory, so $||\beta'(s)|| = 1$ for each s in I. Then $T = \beta'$ is called the unit tangent vector field on β. Since T has constant length 1, its derivative $T' = \beta''$ measures the way the smooth trajectory is turning in (R^3, σ). We call $T'$the curvature vector field of β. Differentiation of $\sigma(T, T) = 1$ gives $\sigma(T, T') = 0$, so T' is always orthogonal to T, that is, normal to β. The length of the curvature vector field T' gives a numerical measurement of the turning of β.
The real-valued map κ such that $\kappa(s) = ||T'(s)||$ for all s in I is called the curvature map of β. Thus $\kappa \geq 0$, and the larger κ is, the sharper the turning of β. To carry this analysis further, we impose the restriction that κ is never zero so $\kappa > 0$. The unit-vector field $N = T'/\kappa$ on β then tells the direction in which β is turning at each point. N is called the principal normal vector field of β. According to Gram-Schmidt process let $\bar{B} = \beta''' - \sigma(\beta', \beta''')\beta' - \kappa^{-2}\sigma(\beta'', \beta''')\beta''$. The vector field $B = ||\bar{B}||^{-1}\bar{B}$ on β is called the binormal vector field of β.

Lemma 3.1. Let β be a unit-speed smooth trajectory in (R^3, σ) with $\kappa > 0$. If $\beta', \beta'', \beta'''$ are linearly independent, then the three vector fields T, N and B on β are unit vector fields that are mutually orthogonal in (R^3, σ) at each point.

Proof. By definition $||T|| = 1$. Since $\kappa = ||T'|| > 0$, $||N|| = 1$. We saw above that T and N are σ orthogonal, that is, $\sigma(T, N) = 0$. If $\beta', \beta'', \beta'''$ are linearly independent, then $\bar{B} \neq 0$, $\sigma(T, \bar{B}) = \sigma(\beta', \beta''') - \sigma(\beta', \beta''')\sigma(\beta', \beta') - \kappa^{-2}\sigma(\beta'', \beta''')\sigma(\beta', \beta'') = 0$ and $\sigma(N, \bar{B}) = \kappa^{-1}\sigma(\beta'', \beta''' - \sigma(\beta', \beta''')\beta' - □^{-2}□(□'', □''')□'') = □^{-1}□(□'', □''') - □^{-1}□(□', □''')□(□', □'') - □^{-3}□(□'', □''')□(□'', □'') = 0$. Therefore, $||□|| = 1$ and B is orthogonal to both □ and □.

Definition 3.2. We call □, □, □ the Frenet–Serret frame field on □ corresponding to □.

Remark 3.3. The key to the successful study of the geometry of a smooth trajectory □ is to use its Frenet–Serret frame field □, □, □ on □ corresponding to □ whenever possible, instead of the standard Frenet–Serret field *T*, *N*, *B*, i.e., Frenet–Serret frame field □, □, □ on □ corresponding to <, >.
The Frenet–Serret frame field of □ corresponding to □ is full of information about □, whereas the standard Frenet–Serret field contains none in $(□^3, □)$ at all , because of its own inner product of the space. The first and most important use of this idea is to express the derivatives $□', □', □'$ in terms of □, □, □. Since $□ = □'$, we have $□' = □'' = □□$. We claim that $□'$ is, at each point, a scalar multiple of □.
To prove this, it suffices by □ orthonormal expansion to show that $□(□', □)$= 0 and $□(□', □) = 0$. The former holds since □ is a unit vector. To prove the latter, differentiate $□(□, □) = 0$, obtaining $□(□', □) + □(□, □')$=0; then $□(□', □) = -□(□, □') = -\sigma(B, \kappa N) = -\kappa\sigma(B, N) = 0$. Thus we can now define the torsion map τ of the smooth trajectory β to be the real-valued map on the interval I such that $B' = -\tau N$.
By contrast with curvature, there is no restriction on the values of τ, it may be positive, negative, or zero at various points of I. We shall presently show that τ does measure the torsion, or twisting, of the smooth trajectory β.

Theorem 3.4. (Frenet–Serret formulas in the general inner product space (R^3, σ)). If $\beta: I \to R^3$ is a unit-speed smooth trajectory in the inner product space (R^3, σ) with curvature κ > 0 and torsion τ, then $T' = \kappa N, N' = -\kappa T + \tau B, B' = -\tau N$.

Proof. As we saw above, the first and third formulas are essentially just the definitions of curvature and torsion. To prove the second, we use σ orthonormal expansion to express N' in terms of T, N, B to find that $N' = \sigma(N', T)T + \sigma(N', N)N + \sigma(N', B)B$. Differentiating $\sigma(N, T) = 0$, we get $\sigma(N', T) + \sigma(N, T') = 0$, hence $\sigma(N', T) = -\sigma(N, T') = -\sigma(N, \kappa N) = -\kappa\sigma(N, N) = -\kappa$. As usual, $\sigma(N', N) = 0$, since N is a unit vector field. Finally, $\sigma(N', B) = -\sigma(N, B') = -\sigma(N, -\tau N) = \tau$.

ARBITRARY SPEED TRAJECTORIES

It is a simple matter to adapt the results of the previous section to the study of a regular smooth trajectory $\alpha: I \to R^3$ that does not necessarily have unit speed. We merely transfer to α the Frenet-Serret apparatus corresponding to σ of a unit-speed reparametrization β of α. Explicitly, if s is an arc length map for α as indicated in Theorem 2.1, then $\alpha(t) = \beta(s(t))$ for all t, or $\alpha = \beta(s)$. Now if $\kappa_\beta > 0, \tau_\beta, T_\beta$, N_β and B_β are defined for β as above, we define for α the curvature map, $\kappa_\alpha(t) = \kappa_\beta$(s); torsion map, $\tau_\alpha(t) = \tau_\beta$(s); tangent vector field, $T_\alpha(t) = T_\beta$(s); normal vector field, $N_\alpha(t) = N_\beta$(s); binormal vector field: $B_\alpha(t) = B_\beta$(s). In general κ_α and κ_β are different maps, defined on different intervals. But they give exactly the same description of the turning of the common route of α and β , since at any point $\alpha(t) = \beta(s(t))$ the numbers $\kappa_\alpha(t)$ and κ_β(s(t)) are by definition the same. The Frenet–Serret formulas corresponding to σ are valid only for unit-speed trajectories; they tell the rate of change of the frame field T, N, B with respect to arc length. However, the speed v of the smooth trajectory is the proper correction factor in the general case.

Lemma 4.1. If α is a regular smooth trajectory in (R^3,σ) with $\kappa > 0$, then $T^{'} = \kappa v N, N^{'} = -\kappa v T + \tau v B, B^{'} = -\tau v N$.

Proof. Let β be a unit-speed reparametrization of α. Then by definition, $T_\alpha(t) = T_\beta$(s), where s is an arc length map for α. The chain rule as applied to differentiation of vector fields gives $T_\alpha^{'}(t) = T_\beta^{'}(s)s^{'}(t)$. By the usual Frenet–Serret equations corresponding to σ, $T_\beta^{'}(s) = \kappa_\beta(s)N_\beta$(s). Substituting the map s in this equation yields $T_\beta^{'}(s) = \kappa_\beta(s)N_\beta(s) = \kappa_\alpha(t)N_\alpha$(t) by the definition of κ_α and N_α in the arbitrary-speed case. Since $s^{'}(t)$ is the speed map v of α, these two equations combine to yield $T_\alpha^{'} = \kappa_\alpha v N_\alpha$. The formulas for $N_\alpha^{'}$ and $B_\alpha^{'}$ are derived in the same way.

Lemma 4.2. If α is a regular smooth trajectory with speed map v in (R^3,σ) with $\kappa > 0$, then the velocity and acceleration of α are given by $\alpha^{'} = vT, \alpha^{''} = v^{'}(t)T + \kappa v^2 N$.

Proof. Since $\alpha = \beta(s)$, where s is the arc length map of α, we find that $\alpha^{'} = \beta^{'}(s)s^{'}(t) = vT_\beta(s) = vT_\alpha(t)$. A second differentiation yields the other.

Example 4.3. Consider (R^3,σ) with inner product $\sigma(p,q) = p_1q_1 + p_2q_2 + 2p_3q_3$. Then $\beta(s) = \left(cos\frac{s}{2}, sin\frac{s}{2}, \sqrt{\frac{3}{8}}s\right)$ unit-speed smooth trajectory, so a simple computation, using Frenet–Serret formulas corresponding to σ shows that

$$T(s) = (-\tfrac{1}{2}sin\tfrac{s}{2}, \tfrac{1}{2}cos\tfrac{s}{2}, \sqrt{\tfrac{3}{8}}),\ N(s) = \left(-cos\tfrac{s}{2}, -sin\tfrac{s}{2}, 0\right),$$
$$B(s) = \left(\tfrac{\sqrt{3}}{2}sin\tfrac{s}{2}, -\tfrac{\sqrt{3}}{2}cos\tfrac{s}{2}, \tfrac{1}{\sqrt{8}}\right), \kappa(s) = \tfrac{1}{4}, \tau(s) = -\tfrac{\sqrt{3}}{4}.$$

Note that the Frenet–Serret formulas for β, as a smooth trajectory in the standard inner product space $(R^3,<,>)$ asserts that

$$T(s) = (-\sqrt{\tfrac{2}{5}}sin\tfrac{s}{2}, \sqrt{\tfrac{2}{5}}cos\tfrac{s}{2}, \sqrt{\tfrac{3}{5}}),\ N(s) = \left(-cos\tfrac{s}{2}, -sin\tfrac{s}{2}, 0\right),$$
$$B(s) = \left(\sqrt{\tfrac{3}{5}}sin\tfrac{s}{2}, -\sqrt{\tfrac{3}{5}}cos\tfrac{s}{2}, \sqrt{\tfrac{2}{5}}\right), \kappa(s) = \tfrac{2}{5}, \tau(s) = \tfrac{\sqrt{6}}{5}.$$

One can see easily that in each case, the formulas of Lemma 4.2 are satisfied.

Example 4.4. The spherical image of a unit-speed smooth trajectory β in (R^3,σ) is the smooth trajectory $\delta = T_\beta = \beta^{'}$ with the same Euclidean coordinates. Thus δ, lies on the unit sphere S, and the motion of δ represents the turning of β. The equation of δ implies that $T_\delta = \delta^{'} = T_\beta{}^{'} = \kappa_\beta N_\beta$, so the speed of δ is equal to the curvature of β, i.e., $v_\delta = \kappa_\beta$. Moreover,$T_\delta^{'} = \delta^{''} = T_\beta{}^{''} = (\kappa_\beta N_\beta)^{'} = \kappa_\beta^{'} N_\beta + \kappa_\beta N_\beta^{'} = -\kappa_\beta^2 T_\beta + \kappa_\beta^{'} N_\beta + \kappa_\beta \tau_\beta B_\beta$. So $T_\delta^{'} = \kappa_\delta v_\delta N_\delta$ implies that $\kappa_\beta^4 + (\kappa_\beta^{'})^2 + \kappa_\beta^2\tau_\beta^2 = \kappa_\delta^2\kappa_\beta^2$, and $\kappa_\delta \geq \sqrt{\kappa_\beta{}^2 + \tau_\beta{}^2}$.

CONFLICT OF INTEREST
There is no conflict of interest.

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