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# CHARACTERISTIC BASED SPLIT SCHEME FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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## ABSTRACT

*In this paper characteristic based split finite element (CBS-FEM), is used for the solution of incompressible flow problems. A remarkable advantage of this method is its capability to solve the compressible and incompressible flow problems for any Reynolds number with the same code. Temporal and spatial discretization of the governing equations in this method is elaborated. And at last two benchmark 2D numerical examples of Navier-Stokes (N-S) equations are used to present the CBS finite element properties and performances. Sensitivity analysis on time step size is also carried out and results are presented*

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KEY WORDS

*characteristic based split; finite element method; Navier-Stokes; CBS*

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## INTRODUCTION

It is known that numerical solutions of incompressible Navier–Stokes equations may suffer from numerical instability due to convective character of the equations often leading to oscillatory solutions if the standard Galerkin procedure is used to discretize the equations, as the standard Galerkin method is only valid for self-adjoint operator equations.

Some methods such as Petrov-Galerkin introduced by Zienkiewicz and coworkers [1], streamline Petrov-Galerkin (SUPG) which is the extension of Petrov-Galerkin in two and three dimensions[2], Taylor-Galerkin presented by Donea that is proved to be the finite element equivalent of the Lax-Wendroff method developed in finite difference context[3],Galerkin least square (GLS) that is a linear combination of standard Galerkin and least square approximations[4] and finite increment calculus (FIC) presented by Onate[5] are developed to overcome the instability due to high convective terms.

Another difficulty arises when incompressible Navier-Stokes equations are encountered is that there is no pressure evolution in continuity equation. One of the very popular procedures of dealing with the pressure terms in incompressible Navier-Stokes equations is the fractional step or projection method initially presented by Chorin in the finite difference context[6,7]. In the projection method, a modified version of the momentum equation in discretized form is first advanced in time to provide an approximation for the velocity field at the next time level. The intermediate velocity field will not, in general, satisfy the divergence-free condition for incompressible flow. The velocity correction or projection step involves the solution of a Poisson equation for pressure (or pressure correction) that is derived from the enforcement of the continuity equation. The pressure correction thus obtained is used to modify the intermediate velocity field. This procedure yields to a mixed formulation, which sometimes restricts the choice of interpolation spaces for the velocity and pressure fields. In finite element context several researchers have used the fractional step method for incompressible flow problems [8-10].

Present study uses characteristic based split finite element for the solution of incompressible flow problems. In what fallows, section 2 states governing equations on fluid flows (Navier-Stokes equations). In section 3, characteristic based split algorithm is explained by two parts termed temporal discretization schema and spatial discretization schema. Section 4 and 5 relates to time restriction criteria and CBS flowchart respectively. And finally in section 6 some numerical examples are described and solved by CBS algorithm.



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# MATERIALS AND METHODS

## Navier-Stokes equations

The Navier-Stokes equations in conservation form may be written as [11]:

### Mass conservation equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} = -\frac{\partial (U_i)}{\partial x_i} \quad (1)$$

Where c is the speed of sound, and $U_i = \rho u_i$ in which $\rho$ is density and $u_i$ is velocity components. It is obvious for incompressible flow speed of sound approaches infinity and the left hand term approaches zero.

### Momentum conservation equation

$$\frac{\partial (U_i)}{\partial t} = -\frac{\partial (u_j U_i)}{\partial x_j} + \frac{\partial (\tau_{ij})}{\partial x_j} - \frac{\partial (p)}{\partial x_i} - \rho g_i \quad (2)$$

Where $\tau_{ij}$ is the deviatoric stress components may obtain by:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) \quad (3)$$

Where $\mu$ is the dynamic viscosity and $\delta_{ij}$ is the kroneker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (4)$$

## Characteristic Based Split Finite Element Method

The CBS scheme is very similar to the original Chorin split or the projection method which is widely employed in incompressible flow calculations. Furthermore it can be used for compressible and incompressible flows. The temporal discretization scheme essentially contains three steps. In the first step, the intermediate velocity field is established. in the second step, the pressure is obtained from continuity equation and finally the intermediate velocities are corrected to get the final velocity values.All three set of equations can be spatially discretized by standard Galerkin procedure [11].

## Temporal discretization

Discretization of the momentum equation (Eq.2) in a typical time interval $[t_n \quad t_{n+1}]$ with $\Delta t = t_{n+1} - t_n$, Using characteristic procedure leads to:

$$U_i^{n+1} - U_i^n = \Delta t\left[-\frac{\partial}{\partial x_j}(u_j U_i)^n + \frac{\partial \tau_{ij}^n}{\partial x_j} + \frac{\partial p^{n+\theta_2}}{\partial x_i} - (\rho g_i)^n\right] + \\ + \frac{\Delta t^2}{2}\left[u_k \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}(u_j U_i) - \frac{\partial p}{\partial x_i} + \rho g_i\right)\right]^n \quad (5)$$

In which $\theta_2 \epsilon [0 \quad 1]$ and $\theta_2$=0,0.5,1 lead to explicit form, Crank-Nicolson semi implicit form and fully implicit form, respectively. An auxiliary variable $U^*$ is introduced in such a way that the characteristic based split of Eq. (5) is written in the form of equations (6) and (7):

$$U_i^* - U_i^n = \Delta t\left[-\frac{\partial}{\partial x_j}\left(u_j U_i\right)^n + \frac{\partial \tau_{ij}^n}{\partial x_j} + \eta\frac{\partial p^n}{\partial x_i} - \left(\rho g_i\right)^n\right] + \frac{\Delta t^2}{2}\left[u_k\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(u_j U_i\right) - \eta\frac{\partial p}{\partial x_i} + \rho g_i\right)^n\right] \quad (6)$$

$$U_i^{n+1} - U_i^* = \Delta t\left[(1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2\Delta t\frac{\partial \Delta p}{\partial x_i} + (1-\eta)\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\frac{\partial p^n}{\partial x_i}\right] \quad (7)$$

Where $\Delta p = p^{n+1} - p^n$. Two types of splitting can be considered by $\eta$ parameter with $\eta = 0$ corresponding to split A (non-iterative splitting scheme) in which all pressure term in momentum equation are splitted and $\eta = 1$ corresponding to split B (iterative splitting scheme) in which only the pressure terms at the $t^{n+1}$ are splitted. Rewriting equation (7) neglecting higher order terms yields to:

$$U_i^{n+1} - U_i^* - U_i^n + U_i^n = \Delta t\left[(1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2\Delta t\frac{\partial \Delta p}{\partial x_i} + (1-\eta)\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\frac{\partial p^n}{\partial x_i}\right] \quad (8)$$

Considering $\Delta U_i^* = U_i^* - U_i^n$, $\Delta U_i = U_i^{n+1} - U_i^n$

$$\Delta U_i = \Delta U_i^* - \Delta t\left[(1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2\Delta t\frac{\partial \Delta p}{\partial x_i}\right] \quad (9)$$

Similarly, the temporal discretization of the continuity equation is written as:

$$\Delta\rho = \left(\frac{1}{c^2}\right)^n \Delta p = -\Delta t\frac{\partial U_i^{n+\theta_1}}{\partial x_i} = -\Delta t\left[\frac{\partial U_i^n}{\partial x_i} + \theta_1\frac{\partial \Delta U_i}{\partial x_i}\right] \quad (10)$$

Considering Eqs. (9) and (10) vanishing higher other terms leads to Eq. (11):

$$\Delta\rho = \left(\frac{1}{c^2}\right)^n \Delta p = -\Delta t\left[\frac{\partial U_i^n}{\partial x_i} + \theta_1\frac{\partial \Delta U_i^*}{\partial x_i} - \Delta t\theta_1\left((1-\eta)\frac{\partial^2 p}{\partial x_i \partial x_j} + \theta_2\frac{\partial^2 \Delta p}{\partial x_i \partial x_j}\right)\right] \quad (11)$$

**Spatial discretization**

The unknown variables U and P are spatially approximated using standard shape functions $N_u$ and $N_p$ as followings.

$$\mathbf{U} = \mathbf{N_u}\bar{\mathbf{U}},\ \mathbf{\Delta U} = \mathbf{N_u}\mathbf{\Delta}\bar{\mathbf{U}},\ \mathbf{\Delta U}^* = \mathbf{N_u}\mathbf{\Delta}\bar{\mathbf{U}}^*,\ p = \mathbf{N_p}\bar{\mathbf{p}},\ \mathbf{u} = \mathbf{N_u}\bar{\mathbf{u}} \quad (12)$$

Using the standard Galerkin procedure, the weak form of equations (6) can be written as:

$$\mathbf{\Delta U}^* = -\mathbf{M_u}^{-1}\Delta t\left[\left(\mathbf{C_u}\bar{\mathbf{U}} + \mathbf{K_r}\bar{\mathbf{u}} + \eta\mathbf{G}^T\bar{p} - \mathbf{f}\right) - \Delta t\left(\mathbf{K_u}\bar{\mathbf{U}} + 0.5\eta\Delta t\mathbf{P}\bar{p} + \mathbf{f_s}\right)\right]^n \quad (13)$$

Where

$$\mathbf{M_u} = \int_\Omega \mathbf{N_u}^T\mathbf{N_u}\,d\Omega \quad (14)$$

$$\mathbf{C_u} = \int_\Omega \mathbf{N_u}^T\left(\nabla\left(\mathbf{u}\mathbf{N_u}\right)\right)d\Omega \quad (15)$$

$$\mathbf{f} = \int_\Omega \mathbf{N_u}^T\rho\mathbf{g}\,d\Omega + \int_\Gamma \mathbf{N_u}^T\mathbf{t^d}\,d\Gamma \quad (16)$$

$$\mathbf{K_r} = \int_\Omega \mathbf{B}^T\mu(\mathbf{I_0} - \frac{\mathbf{2}}{\mathbf{3}}\mathbf{m}\mathbf{m}^T)\mathbf{B}\,d\Omega \quad (17)$$

$$\mathbf{m} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}^T \quad (18)$$

$$\mathbf{g} = \begin{bmatrix}g_1 & g_2 & g_3\end{bmatrix}^T \quad (19)$$

$$\mathbf{I_0} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

$$\mathbf{G} = \int_\Omega (\nabla \mathbf{N}_p)^T \mathbf{N}_u \, d\Omega \quad (21)$$

$$\mathbf{P} = \int_\Omega \left(\nabla\left(\mathbf{u}\mathbf{U}_u\right)\right)^T \nabla \mathbf{N}_p \, d\Omega \quad (22)$$

Spatial discretization of Eq. (11) using a Galerkin method leads to:

$$\int_\Omega N_p^k \Delta\rho \, d\Omega = \int_\Omega N_p^k (\frac{1}{c^2}) \Delta p \, d\Omega = -\Delta t \int_\Omega N_p^k \frac{\partial}{\partial x_i}\left( U_i^n + \theta_1 \Delta U_i^* - \theta_1 \Delta t \left( (1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i} \right)\right) d\Omega \quad (23)$$

With a weak form defined as:

$$\int_\Omega N_p^k \Delta\rho \, d\Omega = \int_\Omega N_p^k (\frac{1}{c^2}) \Delta p \, d\Omega = -\Delta t \int_\Omega \frac{\partial N_p^k}{\partial x_i}\left( U_i^n + \theta_1 \Delta U_i^* - \theta_1 \Delta t \left( (1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i} \right)\right) d\Omega - \Delta t \int_\Gamma N_p^k \left( U_i^n + \theta_1 \Delta U_i^* - \theta_1 \Delta t \left( (1-\eta)\frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i} \right)\right) n_i \, d\Gamma \quad (24)$$

Equation (24) can be shown in matrix form as follows:

$$\left(\mathbf{M_p} + \Delta t^2 \theta_1 \theta_2 \mathbf{H}\right)\Delta\bar{p} = \Delta t\left[\mathbf{G}\bar{\mathbf{U}}^n + \theta_1 \mathbf{G}\Delta\bar{\mathbf{U}}^* - \Delta t \theta_1 \mathbf{H}\bar{\mathbf{p}}^n - \mathbf{f}_p\right] \quad (25)$$

In which

$$\mathbf{H} = \int_\Omega (\nabla \mathbf{N}_p)^T \nabla \mathbf{N}_p \, d\Omega \quad (26)$$

$$\mathbf{M_\rho} = \int_\Omega \mathbf{N}_p^T (\frac{1}{c^2})^n \mathbf{N}_p \, d\Omega \quad (27)$$

$$\mathbf{f}_p = \Delta t \int_\Gamma \mathbf{N}_p^T \mathbf{n}^T \left[\bar{\mathbf{U}}^n + \theta_1\left(\Delta\bar{\mathbf{U}}^* - \Delta t \nabla p^{n+\theta_2}\right)\right] d\Gamma \quad (28)$$

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T \quad (29)$$

And finally equation (9) can also be discretized as:

$$\int_\Omega N_u^k \Delta U_i^{n+1} d\Omega = \int_\Omega N_u^k \Delta U_i^* \, d\Omega - \Delta t \int_\Omega N_u^k \left( \frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i} \right) d\Omega - \frac{\Delta t^2}{2} \int_\Omega \frac{\partial (u_j N_u^k)}{\partial x_j} \frac{\partial p^n}{\partial x_i} d\Omega \quad (30)$$

This equation can be represented in the matrix form as:

$$\Delta\overline{\mathbf{U}} = \Delta\overline{\mathbf{U}}^{*} - \mathbf{M}_u^{-1}\Delta t\left[\mathbf{G}^T(\overline{\mathbf{p}}^{n} + \theta_2\Delta\overline{\mathbf{p}}) + \frac{\Delta t}{2}\mathbf{P}\overline{\mathbf{p}}^{n}\right] \quad (31)$$

# RESULTS

## Stability criteria

This algorithm will always contain an explicit portion in the first characteristic-Galerkin step. However the second step, i.e. that of the determination of the pressure increment, can be made either explicit or implicit and various possibilities exist here depending on the choice of $\theta_2$. Different stability criteria will apply depending on the choice of the parameter $\theta_2$ as zero or non-zero being fully explicit or semi-implicit, respectively.
It is necessary to mention that the fully explicit form is only possible for compressible flow problems for which $c \neq \infty$. In fully explicit form where $0.5 \leq \theta_1 \leq 1$ and $\theta_2 = 0$, the time step limitation is defined as:

$$\Delta t \leq \frac{h}{c + |\mathbf{U}|} \quad (32)$$

as viscosity effects are generally negligible here [11].
The semi-implicit forms defined by $0.5 \leq \theta_1 \leq 1$ and $0.5 \leq \theta_2 \leq 1$ are conditionally stable with the permissible time step size defined by:

$$\Delta t \leq \frac{h}{|\mathbf{U}|} \quad (33)$$

and

$$\Delta t \leq \frac{h^2}{2\nu} \quad (34)$$

Where $h$ is the measure of mesh size and $\nu$ is the kinematic viscosity. Study of Guermond and Quartapelle showed that the splitting methods cannot usually satisfy the LBB (Ladyzhenskaya-Babuska-Brezzi) compatibility condition [12,13]. In the iterative splitting scheme (split B), the velocity-pressure pair must satisfy the LBB condition to obtain non-oscillatory numerical results. By contrast with non-iterative splitting scheme (Split A), equal order interpolations could be safely used, provided the time step is not too small with respect to the spatial mesh size, in the sense that $\Delta t \geq a h^k$, where $k$ is the velocity interpolation order, $h$ a measure of the mesh size and $a$ is a coefficient[14]. Minev presents a discussion on which splitting method requires an LBB compliant approximation and which do not[15].

# DISCUSSION

## CBS algorithm for Navier-Stokes equations

To solve Navier-Stokes equations using CBS, one has to consider following steps:

1- Choosing $\eta$ to select splitting scheme ($\eta = 0$: split A and $\eta = 1$: split B). often split A (non-iterative splitting) is recommended for its property on satisfying LBB compatibility condition.
2- Selecting $\Delta t$ by considering section 4
3- Obtaining $\Delta\overline{\mathrm{U}}^{*}$ from equation (13).
4- Calculating pressure change from equation (25).
5- Computing $\Delta\overline{\mathrm{U}}$ using equation (31).
6- Completing time step calculation.
7- Advancing to the next time step and repeating steps 3 – 6.

## Numerical examples

Two well known examples, the lid-driven cavity and backward facing step are used to demonstrate the capability and performance of the proposed schemes.

## Lid-driven cavity

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As the first example a lid-driven cavity flow problem is considered as shown in **[Figure- 1]**. The top lid of a square and closed cavity (1.0m×1.0m) is assumed to move in its plane with certain uniform prescribed velocity (1.0 m/s). All other walls are assumed to be stationary with zero velocity components imposed on them (no slip walls). Flow is considered laminar and incompressible.

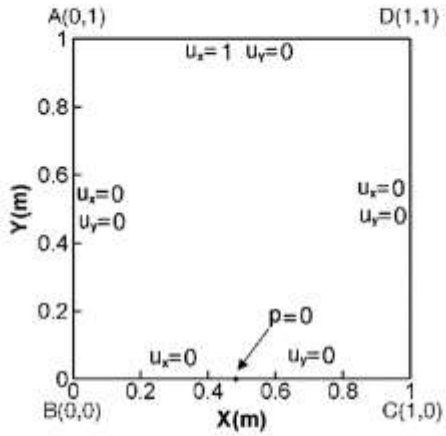


**Fig:1. lid-driven cavity and its boundary conditions**

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The boundary conditions for the velocities are $u_x = 0.0$, $u_y = 0.0$on the boundaries AB, BC, CD and $u_x = 1.0$, $u_y = 0.0$ on the boundary AD and pressure boundary condition is p = 0 at the point E.

Using a mesh, illustrated in **[Figure- 2]**, the steady state solution for the example problem is presented. The problem is solved with three different Reynolds numbers,$Re = 100$,$Re = 500$ and $Re = 5000$ with the characteristic velocity U and characteristic length L used to calculate Reynolds numbers are chosen$1.0\ m/s$ and $1.0\ m$ for this example, respectively. Pressure contours, Streamlines and flow pattern for Re=5000 and$\Delta t = 0.001\ s$ are illustrated in **[Figure- 3 to 5 ]**respectively.

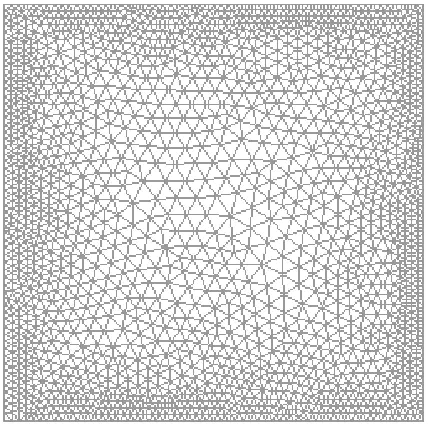

**Fig: 2. Triangular meshing with 3438 elements and 1836 nodes**

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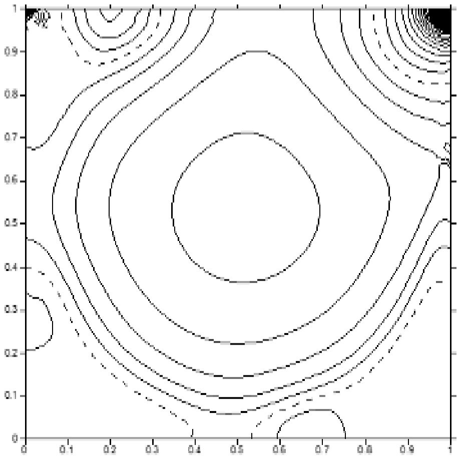

**Fig: 3. P contours for Re=5000**

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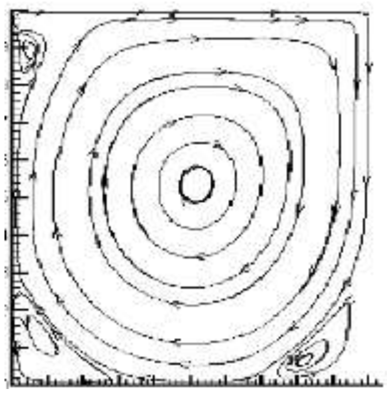

**Fig: 4. Streamline for Re=5000**

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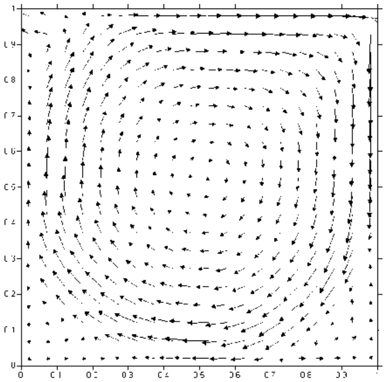

**Fig: 5. Flow pattern for Re=5000**

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U profile at x=0.5 for different Reynolds number (Re=100, Re=500, Re=5000) by$\Delta t = 0.001\ s$ are plotted in **[Figure-6]** Also observed data (by Ghia) and CBS results for Re=5000 are illustrated in **[Figure-7]** Result by CBS and Ghia shows good similarity.

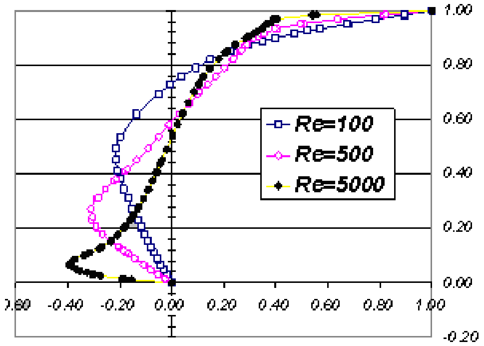


Fig: 6- U profile at x=0.5 for different Reynolds number

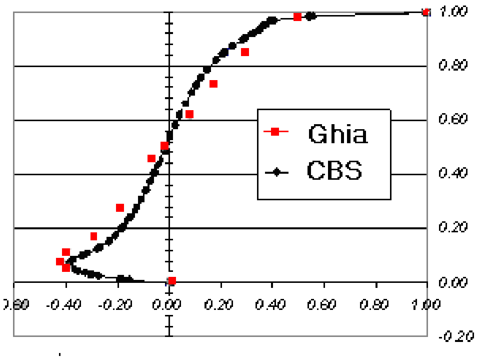


Fig: 7. U profile at x=0.5 with CBS and observed data for Re=5000

To investigate the time increments effect on the steady state solution, the cavity flow problem is solved using the mesh illustrated in [Figure- 8] for Re=100 with four increments $\Delta t = 0.009\ s$, $\Delta t = 0.001\ s$,$\Delta t = 0.0005\ s$ and $\Delta t = 0.00005\ s$. The pressure contours for each$\Delta t$ are illustrated in [Figure-9]. It is obvious that smaller time increments size tend to constitute the oscillating pressure field. As expected very small$\Delta t$, is lead to oscillatory results

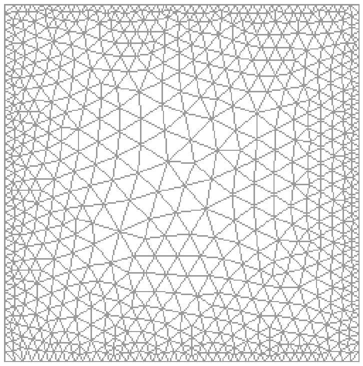

Fig: 8. Mesh with 1392 elements and 778 point

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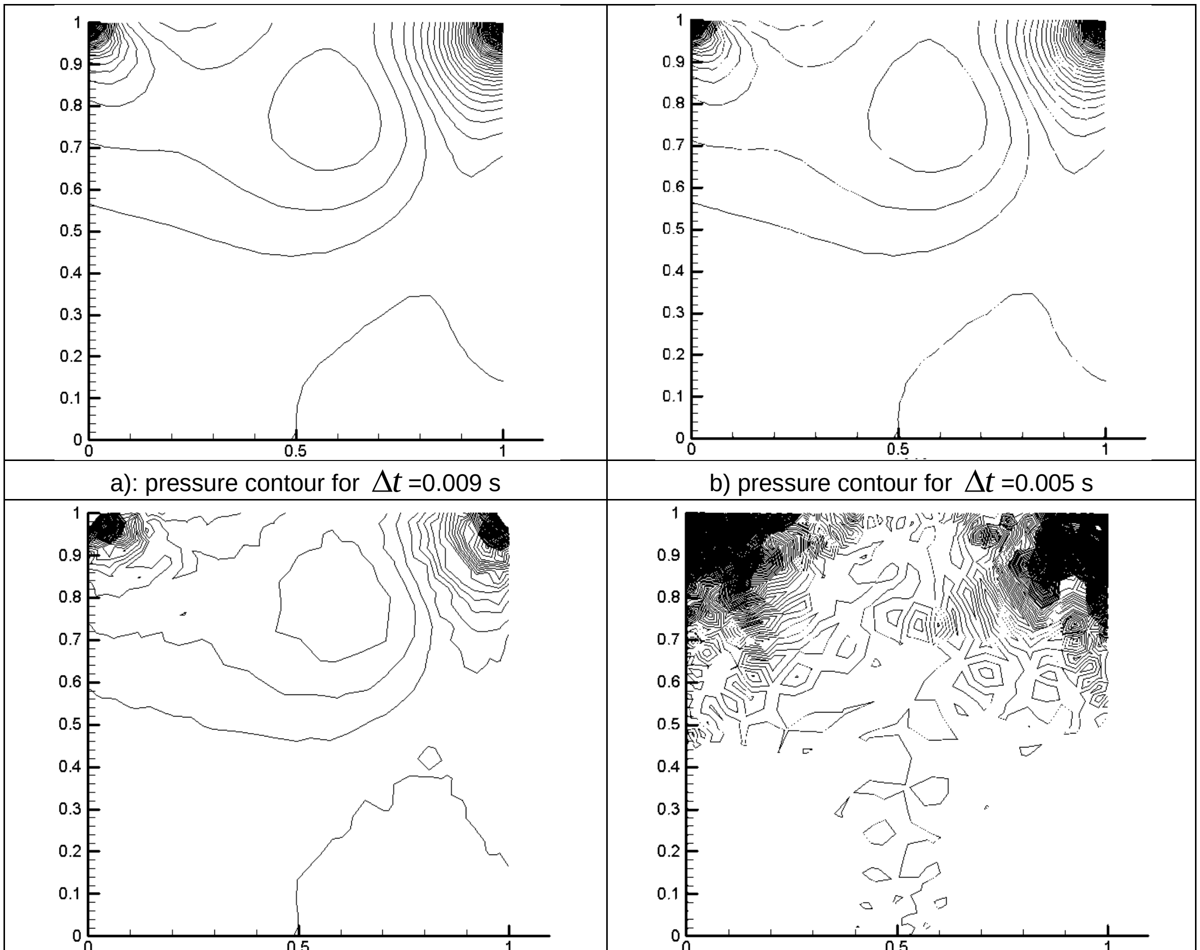

a): pressure contour for $\Delta t$ =0.009 s | b) pressure contour for $\Delta t$ =0.005 s
c) pressure contour for $\Delta t$ =0.0005 s | d) pressure contour for $\Delta t$ =0.00005 s

**Fig: 9. pressure contour for different time increment size**

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### Backward Facing Step

The step has a 4.9mm heights, upstream channel has a length of 19.6 mm and a height of 5.2 mm. Length of channel after step is 196 mm. The boundary condition considered is parabolic horizontal velocity profile with a maximum 1.0 cm/s and at the exit the pressure is prescribed. All solid walls are imposed with no-slip conditions. The mesh is used to solve the problem that is illustrated in **[Figure-10]**.

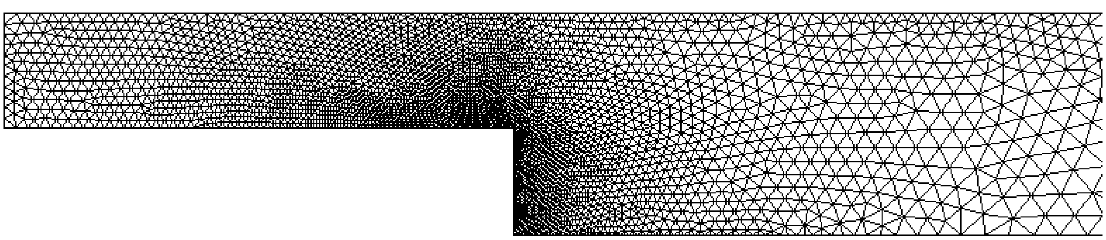

**Fig: 10. Mesh used  for backward facing step**

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A semi implicit scheme with $\theta_1 = \theta_2 = 0.5$ is used here to solve the problem. This problem is solved with Re=100, pressure contours are illustrated in **[Figure-11]** and flow pattern and streamlines are presented in **[Figure-12, 13]**

**Fig: 11. P contours Re=100**

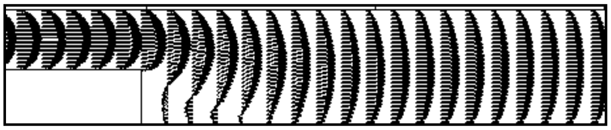

**Fig:12. Flow pattern Re=100**

**Fig:13. Streamline Re=100 around the step**

## CONCLUSION

Characteristic based split finite element method, CBS-FEM, is presented in this paper for Navier-Stokes equations. In this algorithm three steps in each iteration will be done. In the first step an intermediate velocity is calculated from momentum equations with vanishing pressure terms (partially or fully), using an explicit characteristic Galerkin method. At the second step the intermediate velocities are used to compute pressure or pressure increment. Despite fractional step method this step is applicable for both compressible and incompressible flow problems. And finally in the last step, velocity increments are obtained considering intermediate velocities and pressure terms calculated before.

CBS-FEM is used for the solution of the cavity flow problem and backward facing step problem. The results show good similarity with previous works. Furthermore a sensitivity analysis is done on the time increment size and it is concluded that $\Delta t$ must be less than a value due to explicit nature in the first step of CBS-FEM, and it must be greater than a value to satisfy the LBB condition, when the same shape functions are used for the velocity and pressure.

## CONFLICT OF INTEREST

None

## ACKNOWLEDGEMENT

None

## FINANCIAL DISCLOSURE

None

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