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QUANTUM DYNAMICS FOR THE GENERALIZED CALDIROLA-KANAI OSCILLATOR IN COHERENT STATES

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ABSTRACT

A generalization of the Caldirola-Kanai Hamiltonian that describes dissipative systems can be fulfilled by replacing standard exponential function with the q-exponential one. The q-exponential function is a deformed exponential function that can be used, in more than one way, to develop a generalized formalism of statistical dynamics, so-called the non-extensive statistical dynamics. The quantum characteristics of the generalized Caldirola-Kanai oscillator are investigated with the help of a linear invariant operator of the system. It is shown that the eigenstate of the linear invariant operator is closely related to the Glauber coherent state. The fluctuations of position and momentum are illustrated and discussed for the different values of nonextensive parameter q.

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KEY WORDS

invariant operator; Caldirola-Kanai oscillator; coherent state; Hamiltonian; q-exponential function

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[I]INTRODUCTION

Though Boltzmann-Gibbs statistics has attained a remarkable success in theoretical physics, it has turned out that some of dynamical systems do not follow this statistics. As a matter of fact, this difficulty is related to nonextensive features of the system, which requires another statistical formalism. According to this, nonextensive statistical formalism is firstly suggested by Tsallis and, afterwards, successfully applied in many branches of dynamical systems, such as biology [1-3], chaotic systems [4], living systems [5], economics [6], hydrology [7], and nonlinear dissipative dynamical systems [8].

Ozeren has studied the effects of nonextensivity on the Caldirola-Kanai (CK) oscillator [9,10] via destruction and creation operators associated with the simple harmonic oscillator (DCOSHO) [11]. Caldirola-Kanai oscillator is a fundamental model of dissipative systems that is usually used to develop a phenomenological single-particle approach for the damped harmonic oscillator. It is well known that the Hamiltonian of CK oscillator is time-dependent. A useful method to solve quantum solutions of a time-dependent Hamiltonian system is invariant operator method [12-21] that have been firstly introduced by Lewis [12]. There are two kinds of invariant operators for a quadratic (time-dependent) Hamiltonian system, namely, linear invariant operator and quadratic invariant operator [15]. The quadratic invariant operator is broadly adopted by many researchers in studying quantum physics. However, the linear invariant operator is not yet extensively studied and, as a consequence, its properties are scarcely known. Notice that the destruction and creation operators that are necessary in developing quantum theory on the basis of the invariant operator method are different from DCOSHO. The linear invariant operator is useful in studying coherent states while on the other hand Schrödinger solutions based on the quadratic invariant operator correspond to the number states. After Glauber's research on coherence properties of the electromagnetic fields through coherent states [22], the concept of coherent states became important in quantum optics. A wave function in coherent states which follow minimum uncertainty relation is closest to the idealized classical wave. Nonclassical properties of SU(1,1) coherent states for the damped harmonic oscillator are studied in previous papers [23,24]. In the present work, we study the quantum dynamics for the generalized CK oscillator in the Glauber coherent state by employing linear invariant operator method.

[II] MATERIALS AND METHODS

2.1. Linear invariant operator

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The Hamiltonian of the CK oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} e^{-\beta t} + \frac{1}{2} e^{\beta t} m\omega^2 \hat{x}^2, \tag{1}$$

where β is a damping constant. Through Tsallis thermostatics, a generalization of the exponential function appeared in the above equation can be fulfilled by replacing ordinary exponential function with a deformed one known as q-exponential function, such that [25]

$$e^y \to \exp_q y = [1+(1-q)y]^{1/(1-q)}, \tag{2}$$

where $1+(1-q)y > 0.$ As a matter of fact, the q-exponential function is ubiquitous and beautiful. In case that $q \to 1$, $\exp_q y$ reduces to ordinary exponential function. A distinctive feature of q-exponential function we should bear in mind is that $[\exp_q y]^{-1}$ cannot be replaced by $\exp_q(-y)$ except for $q=1.$ In terms of the q-exponential, the CK Hamiltonian can be generalized in the form [11]

$$\hat{H}_q = \frac{\hat{p}^2}{2m\exp_q(\beta t)} + \frac{1}{2}\exp_q(\beta t) m\omega^2 \hat{x}^2. \tag{3}$$

From fundamental dynamics of Hamiltonian, it is not difficult to show that this Hamiltonian yields the classical equation of motion which is represented as

$$\ddot{x}(t) + \frac{\beta}{1+(1-q)\beta t}\dot{x}(t) + \omega^2 x(t) = 0. \tag{4}$$

Let us denote two homogeneous independent real solutions of the above equation as $s_1(t)$ and $s_2(t)$. Then, a little algebra leads to [26]

$$s_1(t) = \left(\frac{\pi\omega}{2\beta(1-q)}\right)^{1/2} \frac{s_0}{[1+(1-q)\beta t]^\nu} J_\nu\left(\frac{\omega}{(1-q)\beta} + \omega t\right), \tag{5}$$

$$s_2(t) = \left(\frac{\pi\omega}{2\beta(1-q)}\right)^{1/2} \frac{s_0}{[1+(1-q)\beta t]^\nu} N_\nu\left(\frac{\omega}{(1-q)\beta} + \omega t\right), \tag{6}$$

where s_0 is a constant, $\nu = q/[2(1-q)]$, J_ν and N_ν are the first and the second kind Bessel functions. The general solution of Eq. (4) is

$$x(t) = c_1 s_1(t) + c_2 s_2(t), \tag{7}$$

where c_1 and c_2 are arbitrary real constants.

Notice that the Hamiltonian, Eq. (3), is dependent on time. It is well known that the quantum solutions of a time-dependent Hamiltonian system can be derived by taking advantage of invariant operators of the system [13]. In general, the invariant operator $\hat{I}$ should satisfy the Liouville-von Neumann equation which is given by

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}_q] = 0. \tag{8}$$

Let us suppose that the solution of the above equation for our system is of the form

$$\hat{I}(\hat{x}, \hat{p}, t) = A(t)\hat{x} + B(t)\hat{p}, \tag{9}$$

where $A(t)$ and $B(t)$ are time functions which should be determined afterwards. The substitution of Eqs (3) and (9) into Eq (8) yields

$$\dot{A} = B\exp_q(\beta t) m\omega^2, \tag{10}$$

$$\dot{B} = -\frac{A}{m\exp_q(\beta t)}. \tag{11}$$

From the above two equations, the differential equation for $B(t)$ is easily derived to be

$$\ddot{B}(t) + \frac{\beta}{1+(1-q)\beta t}\dot{B}(t) + \omega^2 B(t) = 0. \tag{12}$$

Since this is just the same as the classical equation of motion for coordinate given in Eq. (4), we can take the solution of $B(t)$ as $s_1(t)$ or $s_2(t)$. However, in this case, it may be more useful to consider a complex-number solution of the form

$$B(t) = B_0 s(t) \exp\left(i\Omega \int_0^t \frac{dt'}{s^2(t') m \exp_q(\beta t')} \right), \tag{13}$$

where B_0 is an arbitrary complex constant and $s(t)$ and Ω are given by

$$s(t) = \sqrt{s_1^2(t) + s_2^2(t)}, \tag{14}$$

$$\Omega = m[s_1(t)\dot{s}_2(t) - \dot{s}_1(t)s_2(t)]\exp_q(\beta t). \tag{15}$$

From a direct differentiation of Eq. (15) with respect to time, we have $d\Omega/dt = 0$. This implies that Ω is a time constant. Note that the complex conjugate of Eq. (13), $B^*(t)$, is also allowed as a solution of Eq. (12). Let us denote the invariant operator associated with $B(t)$ and $B^*(t)$ as $\hat{I}_I$ and $\hat{I}_{II}$, respectively. By inserting the time derivative of $B(t)$ given in Eq. (13) into Eq. (11), another time function $A(t)$ can also be obtained. Thus, we have

$$\hat{I}_I = \hat{a}\exp\left(i\Omega \int_0^t \frac{dt'}{s^2(t') m \exp_q(\beta t')} \right), \tag{16}$$

$$\hat{I}_{II} = \hat{a}^+ \exp\left(-i\Omega \int_0^t \frac{dt'}{s^2(t') m \exp_q(\beta t')} \right), \tag{17}$$

where

$$\hat{a} = B_0\left[s(t)\hat{p} - \left(m\exp_q(\beta t)\dot{s}(t) + i\frac{\Omega}{s(t)} \right)\hat{x} \right], \tag{18}$$

$$\hat{a}^+ = B_0^*\left[s(t)\hat{p} - \left(m\exp_q(\beta t)\dot{s}(t) - i\frac{\Omega}{s(t)} \right)\hat{x} \right]. \tag{19}$$

If we choose B_0 as

$$B_0 = \left(\sqrt{2\hbar\Omega}\right)^{-1} \exp(i\theta), \tag{20}$$

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where θ is an arbitrary phase, Eqs. (18) and (19) become dimensionless and satisfy the boson commutation relation such that $[\hat{a},\hat{a}^{\dagger}]=1$. Hence, we can conclude that $\hat{a}$ and $\hat{a}^{\dagger}$ are destruction and creation operators, respectively. The advantage of putting the solutions of Eq. (12) as Eq. (13) is that, by doing so, we can easily understand the relations between the linear invariant operators and the ladder operators. Let us take $\theta=\pi/2$ at this stage for convenience. Then, Eqs. (18) and (19) become similar to those of Ref. [26].

2.2. Quantum dynamics

Let us write the eigenvalue equation of $\hat{I}_I$ in the form

$$\hat{I}_I\,|\phi\rangle=\lambda\,|\phi\rangle. \tag{21}$$

If we consider that $\hat{I}_I$ is expressed in terms of $\hat{a}$, the eigenstate $|\phi\rangle$. is associated to the Glauber coherent state. From Eqs. (16) and (21), it is trivial to identify λ, which is

$$\lambda=\alpha\exp\left(i\Omega\int_0^t\frac{dt'}{s^2(t')m\exp_q(\beta t')}\right), \tag{22}$$

where α is an eigenstate of $\hat{a}$. By solving Eq. (21) in position space, we have the eigenstate such that

$$\langle x|\phi\rangle=\left[\frac{\Omega}{s^2\hbar\pi}\right]^{1/4}\exp\left\{\frac{1}{s\hbar}\left[\sqrt{2\hbar\Omega}\alpha x-\frac{1}{2}\left(\frac{\Omega}{s}-im\exp_q(\beta t)\dot{s}\right)x^2\right]-\frac{1}{2}|\alpha|^2-\frac{1}{2}\alpha^2\right\}. \tag{23}$$

According to the invariant operator theory, the wave function, $\langle x|\psi\rangle$, which satisfy the Schrödinger equation is different from the eigenstate of the invariant operator by only a time-dependent phase factor [13]

$$\langle x|\psi\rangle=\langle x|\phi\rangle\exp[i\varepsilon(t)]. \tag{24}$$

To derive the time-dependent phase $\varepsilon(t)$, let us insert Eq. (24) in Schrödinger equation. Then, we get

$$\hbar\dot{\varepsilon}=\langle\phi|\left(i\hbar\frac{\partial}{\partial t}-\hat{H}_q\right)|\phi\rangle. \tag{25}$$

Execution of some algebra after inserting Eq. (3) into the above equation leads to

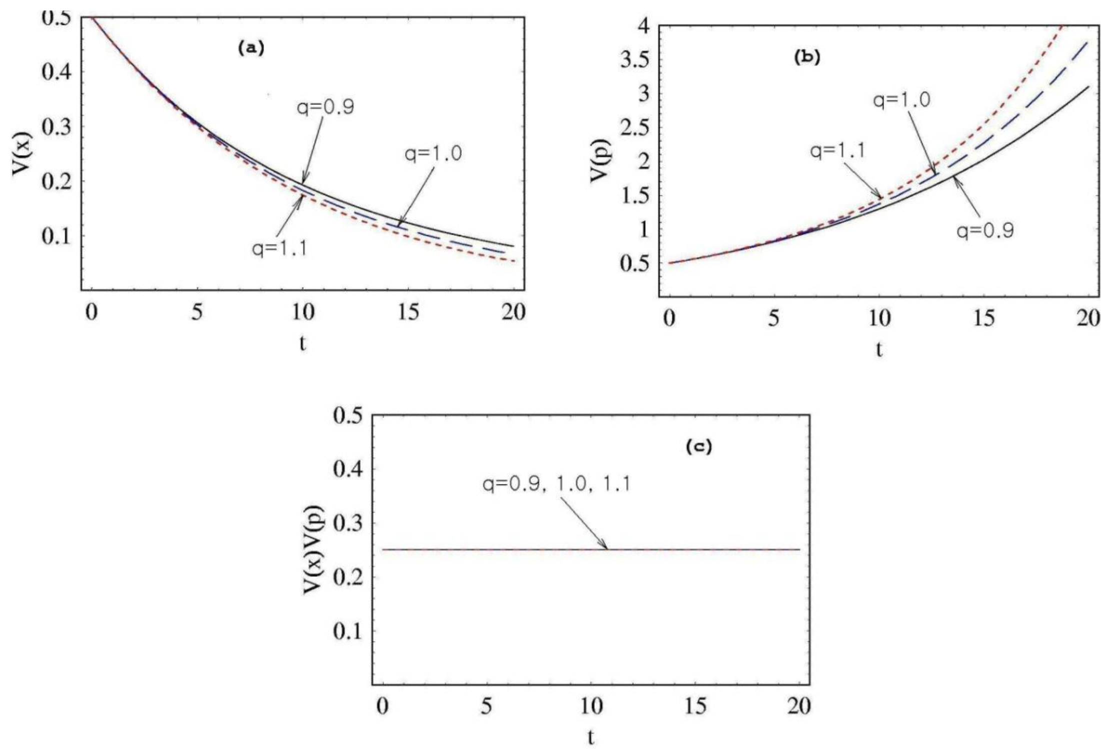

Fig: 1. Variances $V(x)$ (a) and $V(p)$ (b) and uncertainty product $V(x)V(p)$ (c) as a function of time, for various values of q. We used $\omega=1$, $\beta=0.1$, $m=1$, $\hbar=1$, and. $s_0=1$.

$$\dot{\varepsilon}=-\frac{\Omega}{s^2(t)m\exp_q(\beta t)}\left(\alpha^*\alpha+\frac{1}{2}\right)+\frac{i}{2}(\dot{\alpha}\alpha^*-\dot{\alpha}^*\alpha). \tag{26}$$

If we consider that the absolute value of α is constant, $\varepsilon(t)$ in the above equation is easily identified to be

$$\varepsilon(t)=-\frac{1}{2}\Omega\int_0^t\frac{dt'}{s^2(t')m\exp_q(\beta t')}. \tag{27}$$

Thus, we confirm that the exact Schrödinger solution in the coherent state is represented as Eq. (24) with Eqs. (23) and (27). This solution is very useful when we investigate diverse quantum properties of the system.

As an example, let us see the variances of the canonical variables. The variance of an arbitrary quantum variable is in general defined as

$$V(y)=\langle\hat{y}^2\rangle-\langle\hat{y}\rangle^2, \tag{28}$$

where $\langle\hat{y}\rangle=\langle\psi|\,\hat{y}|\psi\rangle$. Considering the wave function given in Eq. (24), the variances of the canonical variables are evaluated in the form

$$V(x)=s^2\hbar/(2\Omega), \tag{29}$$

$$V(p)=\frac{\hbar\Omega}{2s^2}\left[1+\left(\frac{m\exp_q(\beta t)s\dot{s}}{\Omega}\right)^2\right]. \tag{30}$$

Variances $V(x)$ and $V(p)$ and their product (uncertainty product) $V(x)V(p)$ are illustrated in **Fig. 1**. Note that $V(x)$ decreases as time goes by while $V(p)$ increases. However, the uncertainty product $V(x)V(p)$ almost does not vary with time and is nearly $\hbar^2/4$. The uncertainty product in the coherent state is the same as the minimum uncertainty product in the number state, which corresponds to the case of $n=0$ where n is the quantum number. The decrease of $V(x)$ and the increase of $V(p)$ with time are more rapid for large q.

The degree of correlation between the two conjugate canonical variables x and p is represented by correlation coefficient that is defined as [27].

$$r=\frac{1}{2}\langle\{\hat{x}-\langle\hat{x}\rangle,\,\hat{p}-\langle\hat{p}\rangle\}_+\rangle\Big/\sqrt{V(x)V(p)}, \tag{31}$$

where $\{\ \}_+$ means anti-commutator. A little algebra using Eqs. (24), (29), and (30) leads to

$$r=\frac{1}{\Omega}m\exp_q(\beta t)s\dot{s}\left[1+\left(\frac{m\exp_q(\beta t)}{\Omega}\right)^2\right]^{-1/2}. \tag{32}$$

Then, the uncertainty product $V(x)V(p)$ in the Glauber coherent state for the time-dependent Hamiltonian system is represented in terms of r as [14]

$$V(x)V(p)=\hbar^2/[4(1-r^2)]. \tag{33}$$

This becomes large as r approaches unity. We see from Eq. (32) that r is close to unity when $\dot{s}$ is sufficiently higher than $\Omega/[m\exp_q(\beta t)s]$, leading to high uncertainty product.

[III] RESULTS AND DISCUSSION

The ordinary CK oscillator has been generalized by replacing standard exponential function with the q-exponential function. The quantum dynamics of the generalized CK oscillator is investigated by introducing linear invariant operator. By means of the Liouville-von Neumann equation, two kinds of linear invariant operator $\hat{I}_I$ and $\hat{I}_{II}$ are constructed. We confirmed that $\hat{I}_I$, and $\hat{I}_{II}$ are represented in terms of the destruction operator $\hat{a}$ and the creation operator $\hat{a}^+$, respectively. By solving the eigenvalue equation of $\hat{I}_I$, the corresponding eigenstate $\langle x|\phi\rangle$ has been obtained as shown in Eq. (23). From Eq. (24), we can see that the solution of the Schrödinger equation $\langle x|\psi\rangle$ can be represented in terms of the eigenstate $\langle x|\phi\rangle$. Since $\langle x|\phi\rangle$ is not only an eigenstate of $\hat{I}_I$ but also an eigenstate of $\hat{a}$, we can conclude that Eq. (24) is the Glauber coherent state.

The Schrödinger solution, Eq. (24), is very useful when investigating diverse properties of the coherent state for the generalized CK oscillator. As an example, we evaluated variances of position and momentum by taking advantage of Eq. (24). It is shown in **Fig. 1** that $V(x)$ decreases with time while $V(p)$ increases. The degree of these decrease and increase is explicitly dependent on the nonextensive parameter q. The strength of correlation for position and momentum can be represented by correlation coefficient r defined in Eq. (31). We confirmed that there is a simple relation between $V(x)$ $V(p)$ and r, which is given in Eq. (33).

In recent years, it turned out that nonextensive formalism of thermodynamics with q-exponential is important as an implement of biophysics. A universal function for the kinetics of complex biological systems, which unifies and generalizes several theoretical attempts for describing biological fractal phenomena, has been established with the use of nonextensive mechanics [2]. Useful informations for enzyme−ligand fluorescence energy transfer can be obtained from Tsallis statistics for fluorescence intensity decay in enzyme-ligand complex formation [3].

[VI] CONCLUSION

The time behavior of variances, $V(x)$ and $V(p)$, is affected by the degree of nonextensivity that is determined by the value of q. Fundamentally, $V(q)$ decreases with time while $V(p)$ increases, in response to the dissipation of quantum energy of the oscillator. The rate of decrease of $V(x)$ and the rate of increase of $V(p)$ become large along the increase of q. For q=1, these dynamical behaviors recover to those of standard CK oscillator. The effects of nonextensivity on some dynamical systems are important as a correction of Boltzmann-Gibbs statistics.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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